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Équations d'isomonodromie, solutions algébriques et dynamique.

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[1] En espérant que la réciproque soit vraie.

[2] Même si Guy est venu de moins loin.

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[3] Par ordre décroissant d'étage pour les connaisseurs
[4] Flambant neuve !
[5] Envers qui j'ai une ardoise de café déraisonnable.
[6] Non contractuel !
[7] Figurativement ou littéralement selon le cas.
[8] En particulier, si j'ai bien compris, mes parents.

Contents

Résumé en français

Cette thèse est dédiée à la construction de solutions algébriques d'équations d'isomonodromie et à l'étude de différents procédés effectifs pour générer et calculer de tels objets. Les travaux présentés ici s'articulent autour de plusieurs correspondances établies ces dernières décennies entre des objets de nature analytique (solutions de systèmes hamiltoniens) et géométrique (orbites sous une certaine action du groupe modulaire), ces dernières nous permettant d'utiliser des outils provenant de diverses branches des mathématiques pour parvenir à nos fins.

Si E est un fibré vectoriel au–dessus d'une variété complexe X, une connexion logarithmique ∇ sur E est un morphisme $\mathbb{C}$–linéaire entre le faisceau des sections de E et le produit tensoriel de ce dernier par celui des 1–formes méromorphes à pôles logarithmiques sur X vérifiant une formule de Leibnitz. Une telle connexion est dite plate si elle admet un système fondamental de sections horizontales (*i.e* dans le noyau de ∇) en tout point du complémentaire de son lieu polaire dans X. Le prolongement analytique de telles sections livre une représentation ρ_∇ du groupe fondamental complémentaire du lieu polaire de ∇ dans X, appelée monodromie de la connexion.

Une déformation isomonodromique algébrique sur X est une famille algébrique de fibrés vectoriels sur X munis de connexions logarithmiques plates de même (modulo conjugaison à l'arrivée) représentation de monodromie. Il a été établi par Schlesinger, Garnier et Malmquist que ces objets sont équivalents à la donnée de solutions algébriques d'une famille particulière d'équation aux dérivées partielles, appelées systèmes de Garnier. Les solutions générales de ces systèmes sont mal connues, et l'objectif principal des travaux présentés ici est de construire explicitement de telles déformations isomonodromiques afin d'obtenir de nouvelles solutions algébriques.

Plus particulièrement, considérons une courbe Q (non nécessairement irréductible ou lisse) projective complexe de degré cinq dans le plan projectif $\mathbb{P}^2(\mathbb{C})$. Si l'on dispose d'une connexion logarithmique plate ∇ de rang 2 au–dessus de $\mathbb{P}^2(\mathbb{C})$ dont le lieu polaire est égal à Q, alors pour toute droite générique $L \subset \mathbb{P}^2$, la connexion ∇_L obtenue par restriction de ∇ à L peut être assimilée à une connexion logarithmique plate au–dessus de la droite projective complexe $\mathbb{P}^1(\mathbb{C})$ dont le lieu polaire est égal à cinq points distincts de cette dernière ; de plus la platitude de ∇ permet de prouver que la famille des connexions ∇_L forme une déformation isomonodromique paramétrée par un ouvert de Zariski dans le dual de $\mathbb{P}^2$. L'objet de la première partie de cette thèse est de classifier les solutions algébriques

de systèmes de Garnier pouvant être obtenues par ce procédé.

Nous commençons par déterminer pour quelles courbes quintiques planes dans $\mathbb{P}^2(\mathbb{C})$ il est possible d'obtenir une déformation isomonodromique algébrique n'appartenant pas aux familles d'exemples déjà construites par Mazzocco [46] et Diarra [24] ; nous utilisons pour ce faire la classification des groupes fondamentaux de complémentaires de telles courbes quintiques établie par Degtyarev [19]. Il est suffisant de mener cette étude au niveau des représentations de ces groupes fondamentaux ; en effet, la correspondance de Riemann–Hilbert classique [21] affirme en particulier qu'à toute telle représentation il est possible de faire correspondre une connexion logarithmique plate. Dans un second temps, nous construisons de façon explicite les déformations isomonodromiques et solutions algébriques de Garnier associées à ces représentations de groupes.

Le deuxième volet de cette étude concerne la dynamique de l'action du groupe modulaire d'une sphère épointée sur la variété des caractères associée. Plus précisément, comme le groupe fondamental d'une sphère à r trous est isomorphe à un groupe libre à $r-1$ générateurs $\mathbf{F}_{r-1}$, une représentation de ce groupe dans $SL_d(\mathbb{C})$ est totalement décrite par un r–uplet de matrices dont le produit est égal à l'identité ; on en déduit une action par conjugaison diagonale de $SL_d(\mathbb{C})$ sur la variété des représentations de ce groupe fondamental. Le quotient catégorique (au sens de la théorie géométrique des invariants) de cet espace par cette action est appelé variété des caractères de $\mathbf{F}_{r-1}$ dans $SL_d(\mathbb{C})$ et noté $\mathrm{Char}_d(0,r)$. Comme le groupe modulaire $\mathrm{Mod}(0,r)$ formé par les classes d'isotopie des homéomorphismes de la sphère à r trous agit par automorphismes extérieurs sur le groupe fondamental de cette dernière, on obtient une action de $\mathrm{Mod}(0,r)$ sur $\mathrm{Char}_d(0,r)$. Les travaux de Dubrovin, Mazzocco [27] et Cousin [16] ont établi une correspondance entre les solutions algébriques de systèmes de Garnier et les orbites finies sous cette action. Nous calculons explicitement les orbites associées aux solutions obtenues par le procédé décrit ci–dessus et montrons que ce dernier donne naissance à deux familles à paramètres de solutions algébriques distinctes. Nous détaillons une méthode effective pour calculer des telles orbites à l'aide d'outils de calcul formel.

Dans la deuxième partie de cette thèse, nous étudions le procédé de convolution intermédiaire de Katz et quelques–unes de ses applications à l'étude des déformations isomonodromiques algébriques de surfaces complexes. Ce procédé utilise l'action naturelle du groupe des tresses à r brins d'Artin sur le groupe libre à $r-1$ générateurs pour construire une application sur la variété

$$\mathrm{Char}_*(g,b) \ := \bigcup_{d\in\mathbb{N}^*} \mathrm{Char}_d(0,r)$$

équivariante sous l'action du groupe modulaire $\mathrm{Mod}(0,r)$. Nous inspirant des travaux de Boalch [7] et utilisant la description explicite de la convolution intermédiaire donnée par Völklein [57], Dettweiler et Reiter [22, 23], nous construisons de nouvelles orbites finies sous l'action de ce dernier groupe. Plus précisément, partant de représentations dont les images sont contenue dans un sous–groupe fini de $SL_3(\mathbb{C})$, nous obtenons de nouveaux morphismes de $\mathbf{F}_4$ dans $SL_2(\mathbb{C})$ dont les orbites sous l'action du groupe modulaire doivent être finies par équivariance. Un algorithme explicite pour mener à bien ces calculs est présenté.

Enfin, nous nous intéressons à un possible analogue de cette convolution intermédiaire dans le cas d'un tore à deux trous $\mathbb{T}_2^2$. En l'occurrence, nous définissons un procédé équivariant sous l'action des automorphismes extérieurs du groupe fondamental d'un tore à trois trous envoyant une représentation de $\pi_1(\mathbb{T}_2^2)$ sur une représentation définie sur un de ses sous–groupes distingués. Nous donnons un algorithme explicite mettant en œuvre cette nouvelle convolution intermédiaire.

I1 Déformations isomonodromiques, systèmes de Garnier et dynamique du groupe modulaire

Dans la suite de cette introduction, nous définissons les concepts nécessaires à l'établissement des résultats que nous venons d'énoncer, en donnant les éléments de contexte historique pertinents. Une fois ceux–ci établis, nous énonçons précisément les résultats importants de cette thèse.

I1.1 Déformations isomonodromiques de sphères épointées

I1.1.1 Connexions logarithmiques plates

On se fixe dans ce paragraphe une variété analytique complexe X et un fibré vectoriel $E \to X$ de rang r sur X. On notera $\mathcal{O}_X$ (resp. $\mathcal{M}_X$) le faisceau des fonctions holomorphes (resp. méromorphes) sur X et Ω^1_X (resp. $\mathcal{M}^1_X$) celui des 1–formes holomorphes (resp. méromorphes) sur X. Pour tout fibré vectoriel $F \to X$ au–dessus de X on notera $\Gamma(\cdot, F)$ (resp. $\mathcal{M}(\cdot, F)$) le faisceau des sections holomorphes (resp. méromorphes) de F. Pour une exposition plus complète des résultats présentés ici, nous renvoyons le lecteur à la référence [50].

Définition et écriture locale.

Définition I1.1. *On appelle* connexion méromorphe *sur E tout morphisme $\mathbb{C}$–linéaire*

$$\nabla \ : \Gamma(\cdot, E) \to \mathcal{M}^1_X \otimes_{\mathcal{O}_X} \Gamma(\cdot, E)$$

tel que pour toute section locale (f, s) de $\mathcal{O}_X \times E$ on ait l'identité de Leibniz :

$$\nabla(f \cdot s) = \mathrm{d}f \cdot s + f \cdot \nabla s \ . \tag{E1}$$

Ceci implique que dans une trivialisation locale de E on peut écrire

$$\nabla = \mathrm{d} + \Omega \tag{E2}$$

avec Ω une matrice de 1–formes méromorphes locales sur X. Si pour toute telle écriture Ω et $\mathrm{d}\Omega$ sont à pôles simples, on dira que ∇ est une *connexion logarithmique* sur E.

Ces expressions locales permettent de "visualiser" une telle connexion comme un système différentiel : en effet, au–dessus d'un ouvert de trivialisation $U \subset X$ du fibré E, rechercher les *sections horizontales* de ∇, *i.e* les éléments $s \in \Gamma(U, E)$ tels que $\nabla s = 0$, revient à chercher les solutions $Y\ : U \to \mathbb{C}^r$ du système différentiel :

$$\mathrm{d}Y = -\Omega Y\ . \tag{E3}$$

Ceci nous permettra aussi de parler des *résidus de la connexion* ∇, que nous définissons comme ceux d'une telle matrice Ω au voisinage du pôle considéré.

Définition I1.2. *Soient ∇ et ∇' deux connexions sur le fibré vectoriel E et soit U un ouvert de trivialisation de E sur lequel on ait les écritures locales suivantes :*

$$\nabla = \mathrm{d} + \Omega \quad \textit{et} \quad \nabla' = \mathrm{d} + \Omega'\ .$$

1. *On dit que ∇ et ∇' sont* jauges–équivalentes *sur U s'il existe une application holomorphe $H\ : U \to GL_r(\mathbb{C})$ telle que :*

$$\Omega' = H\Omega H^{-1} - \mathrm{d}H \cdot H^{-1} \quad ;$$

2. *on dit que ∇ et ∇' sont (partout)* jauges–équivalentes *si elles le sont sur tout ouvert de trivialisation de E.*

Définition I1.3. *Une connexion méromorphe sur le fibré vectoriel $E \to X$ est dite* plate *si elle admet un système fondamental de solutions en tout point, i.e si pour tout $x_0 \in X$ il existe un voisinage V de x_0 et r sections locales linéairement indépendantes $s_1, \dots, s_r \in \Gamma(V, E)$ telles que :*

$$\forall i \in [r] := \{1, \dots, r\},\ \nabla s_i = 0\ .$$

Remarque I1.4. 1. *Ceci entraîne que, quitte à intersecter V avec un ouvert de trivialisation de E de façon à ce que ∇ y admette une écriture locale du type (E2), on a équivalence entre les propriétés suivantes :*

(i) *∇ est plate ;*

(ii) *pour tout $x_0 \in X$ n'appartenant pas au lieu polaire de ∇, il existe une unique matrice fondamentale pour ∇ en x_0, i.e il existe un voisinage de trivialisation*

V de x_0 et une unique application holomorphe $B : V \to GL_r(\mathbb{C})$ tels que :

$$\begin{cases} \mathrm{d}B = -\Omega B \\ B(x_0) = I_r \end{cases} ;$$

(iii) on a l'égalité de 1–formes méromorphes suivante au voisinage de x_0 :

$$\mathrm{d}\Omega = \Omega \wedge \Omega \quad ; \tag{E4}$$

(iv) ∇ est localement jauge–équivalente à la connexion triviale d.

2. *La propriété (iii)* supra *implique en particulier que si* $\dim(X) = 1$ *alors toute connexion méromorphe y est plate. De plus, si $r = 1$, une connexion est plate si et seulement la 1–forme méromorphe définie par Ω est une forme fermée.*

La notion d'équivalence de jauge permet en pratique de donner des "modèles locaux" des connexions étudiées. Par exemple, si ∇ est une connexion logarithmique plate dont le lieu polaire est une hypersurface lisse $Y \subset X$, on peut munir X d'un système de coordonnées holomorphes locales $(x_1, \dots, x_n)$ dans lesquelles $Y = \{x_1 = 0\}$ et alors ∇ est localement jauge–équivalente à une connexion du type suivant [50] :

$$\mathrm{d} + M(x_1)\frac{\mathrm{d}x_1}{x_1} \ ,$$

où M est une application holomorphe locale en x_1 à valeurs dans $\mathcal{M}_r(\mathbb{C})$ appelée *résidu* de ∇ en Y.

Monodromie. Supposons que ∇ soit une connexion logarithmique plate sur E dont le lieu polaire soit égal à une certaine hypersurface (non nécessairement lisse) $Y \subset X$ (*i.e* telle que $\nabla_{|X \setminus Y}$ soit holomorphe). Fixons un point $x_0 \in X \setminus Y$ et considérons un lacet continu $\gamma : [0, 1] \to X \setminus Y$ basé en x_0. Comme ∇ est plate, il est possible de considérer l'unique matrice fondamentale B associée en x_0 ; cette dernière étant holomorphe au voisinage de x_0, on peut la prolonger analytiquement le long de γ. *In fine*, on obtient une application holomorphe B^γ définie au voisinage de x_0 à valeurs dans $GL_r(\mathbb{C})$ telle que $\nabla B^\gamma = \nabla B = 0$. Cependant, $B^\gamma(x_0)$ n'est pas nécessairement égale à l'identité.

Proposition I1.5. [50] *Le procédé décrit ci–dessus fournit une représentation de groupes, appelée* monodromie *de la connexion ∇ :*

$$\begin{aligned} \rho_\nabla \ : \pi_1(X \setminus Y, x_0) &\to GL_r(\mathbb{C}) \\ [\gamma] &\mapsto B^\gamma \end{aligned} \quad .$$

Remarque I1.6. *On a la propriété suivante* [50] *: si ∇ et ∇' sont deux connexions logarithmiques plates jauge–équivalentes sur E ayant le même lieu polaire $Y \subset X$, alors il* *existe une matrice $M \in GL_r(\mathbb{C})$ telle que*

$$\forall [\gamma] \in \pi_1(X \setminus Y, x_0), \quad \rho_\nabla([\gamma]) = M \cdot \rho_{\nabla'}([\gamma]) \cdot M^{-1} .$$

I1.1.2 Équations de Schlesinger

Dans toute la suite, on se fixe un fibré vectoriel de rang deux $E^0 \to \mathbb{P}^1(\mathbb{C})$ (la droite projective étant identifiée via une coordonnée *ad–hoc* à $\mathbb{C} \cup \{\infty\}$) muni d'une connexion logarithmique plate ∇^0 ayant ses pôles en n points distincts $a_1^0, \ldots, a_n^0 \in \mathbb{C}$ et au point à l'infini.

Supposons de plus que ∇^0 soit une $\mathfrak{sl}_2(\mathbb{C})$–connexion, *i.e* que sa monodromie ρ^0 soit à valeurs dans $SL_2(\mathbb{C})$ (et donc que les résidus de ∇^0 en ses pôles soient dans $\mathfrak{sl}_2(\mathbb{C})$). On se pose ensuite la question suivante : *comment est–il possible de déformer les résidus de* ∇^0 *(vus comme fonctions de* $\underline{a}^0 := (a_1^0, \ldots, a_n^0)$*) de telle sorte que la monodromie reste inchangée (à conjugaison près) ?*

Déformations isomonodromiques. Commençons par fixer un espace de paramètres adéquat en considérant le revêtement universel $\tilde{Z} \xrightarrow{\pi} Z$ de

$$Z := \{\underline{a} \in \mathbb{C}^n \,|\, \forall i \neq j,\, a_i \neq a_j\}$$

sur lequel on dispose des projections naturelles

$$\begin{aligned} \tilde{\mathrm{pr}}_i \ : \tilde{Z} &\to \mathbb{C} \\ \underline{a} &\mapsto (\pi(\underline{a}))_i \end{aligned}$$

pour $i \in [n]$. On a alors le résultat suivant.

Théorème I1.7 (Malgrange, 1983 [44])**.** *À isomorphisme près, il existe un unique fibré à connexion (E, ∇) de rang 2 au–dessus de $\mathbb{P}^1 \times \tilde{Z}$ tel que :*

(i) ∇ est une $\mathfrak{sl}_2(\mathbb{C})$–connexion logarithmique plate dont les pôles sont exactement les

$$Y_i := \{(x, \underline{a}) \in \mathbb{P}^1 \times \tilde{Z} \,|\, x = \tilde{\mathrm{pr}}_i(\underline{a})\} \text{ pour } i \in [n]$$

et

$$Y_\infty := \{(\infty, \underline{a}) \,|\, \underline{a} \in \tilde{Z}\} \ ;$$

(ii) pour tout $\underline{a} \in \tilde{Z}$, si on considère l'injection

$$\begin{aligned} i \ : \mathbb{P}^1 &\hookrightarrow \mathbb{P}^1 \times \tilde{Z} \\ x &\mapsto (x, \underline{a}) \end{aligned}$$

alors i^*E est isomorphe comme fibré vectoriel à E^0 et $i^*\nabla$ est jauge–équivalente à ∇^0. En particulier, les monodromies de ces deux connexions sont conjuguées.

Le fibré à connexion (E, ∇) est appelé déformation isomonodromique universelle *de (E^0, ∇^0).*

Équations d'isomonodromie. Supposons à présent que les fibrés vectoriels E^0 et E soient triviaux et que le résidu de ∇ le long de Y_∞ soit une fonction constante de la variable $\underline{a} \in \tilde{Z}$; on a alors une écriture (globale) de la forme suivante [50] :

$$\nabla = \mathrm{d} + \sum_{i=1}^{n} A_i(\underline{a}) \frac{\mathrm{d}(x - a_i)}{x - a_i}$$

avec l'abus de notation $a_i := \tilde{\mathrm{pr}}_i(\underline{a})$. En remplaçant ceci dans l'équation (E4) exprimant la platitude de ∇ on montre alors le résultat suivant.

Théorème I1.8 (Schlesinger)**.** *Avec les notations* supra*, on a l'équivalence entre les propriétés suivantes :*

(i) ∇ *est plate ;*

(ii) les résidus de ∇ vérifient les équations de Schlesinger *:*

$$\forall i \in [n], \quad \mathrm{d}A_i = -\sum_{j \neq i} \frac{\{A_i, A_j\}}{a_i - a_j} \mathrm{d}(a_i - a_j)$$

où $\{A_i, A_j\} := A_i A_j - A_j A_i$.

Supposons que A_∞ soit diagonalisable ; alors quitte à changer de jauge on peut supposer

$$A_\infty = \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}.$$

Ceci implique que le coefficient $(2, 1)$ de la matrice

$$A := \sum_{i=1}^{n} \frac{A_i(\underline{a})}{x - a_i}$$

est de la forme

$$\frac{P(x, \underline{a})}{(x - a_1) \dots (x - a_n)}$$

avec $P(\cdot, \underline{a})$ un polynôme en x de degré $n - 2$. Les racines de ce dernier donnent alors $n - 2$ fonctions des variables $a_1, \dots, a_n$. Quitte à composer par une homographie ι envoyant (a_{n-2}, a_{n-1}) sur $(0, 1)$ et fixant le point à l'infini, on obtient $n - 2$ fonctions $r_1, \dots, r_{n-2}$ (données par les racines de $P(\cdot, \iota(\underline{a}))$) dépendant de $n - 2$ variables indépendantes.

I1.2 Systèmes de Garnier

Nous sommes maintenant en mesure d'énoncer un résultat fondamental liant déformations isomonodromiques de sphères épointées et solutions algébriques d'une classe particulière de systèmes hamiltoniens, en l'occurrence les sytèmes de Garnier. Pour définir un tel système, commençons par considérer, pour $n \geq 1$:

$$\left\{\begin{array}{lll} \partial_{s_k}\nu_i & = -\partial_{\rho_i}K_k & i,k \in [n] \\ \partial_{s_k}\rho_i & = \partial_{\nu_i}K_k & i,k \in [n] \end{array}\right. , \tag{E5}$$

en les inconnues $(\nu_i, \rho_i) = (\nu_i(\underline{s}), \rho_i(\underline{s}))$. Les fonctions K_k sont ici de la forme :

$$K_k := -\frac{\Lambda(s_k)}{T'(s_k)} \sum_{i=1}^{n} \frac{T(\nu_i)}{(\nu_i - s_k)\Lambda(\nu_i)} \left(\rho_i^2 - \sum_{j=1}^{n+2} \frac{\theta_j - \delta_{k,j}}{\nu_i - s_j}\rho_i + \frac{\kappa}{\nu_i(\nu_i - 1)} \right) ,$$

$$\text{où} \quad \left\{\begin{array}{ll} \kappa & := \dfrac{1}{4}\left(\left(\sum_{j=1}^{n+2} \theta_j - 1 \right)^2 - (\theta_\infty + 1)^2 \right) , \\ \Lambda & : u \mapsto \prod_{j=1}^{n}(u - \nu_j) , \\ T & : u \mapsto \prod_{j=1}^{n+2}(u - s_j) , \end{array}\right.$$

avec la convention $s_{n+1} = 0$, $s_{n+2} = 1$ et pour un $n+3$–uplet de paramètres fixés $(\theta_1, \ldots, \theta_{n+2}, \theta_\infty) \in \mathbb{C}^{n+3}$. Effectuons à présent le changement de variables suivant :

$$\left\{\begin{array}{ll} t_k & := \dfrac{s_k}{s_k - 1} , \\ \mathbf{q}_i & := s_i \dfrac{\Lambda(s_i)}{T'(s_i)} , \\ \mathbf{p}_i & := (1 - s_i) \sum_{k=1}^{n} \dfrac{T(s_i)\rho_k}{\Lambda'(\nu_k)\nu_k(\nu_k - 1)(\nu_k - s_i)} , \end{array}\right.$$

qui transforme (E5) en un nouveau système hamiltonien [46], que nous appellerons système de Garnier $(\mathcal{G}_n)$:

$$\left\{\begin{array}{lll} \partial_{t_k}\mathbf{p}_i & = -\partial_{\mathbf{q}_i}H_k & i,k \in [n] \\ \partial_{t_k}\mathbf{q}_i & = \partial_{\mathbf{p}_i}H_k & i,k \in [n] \end{array}\right. . \tag{E6}$$

Dans le cas d'une seule variable ($n = 1$), ce changement de paramètres n'est pas nécessaire : le système (E5) vérifie également la propriété de Painlevé et est équivalent à l'équation de Painlevé VI :

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}u^2} = & \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-u} \right) \left(\frac{\mathrm{d}y}{\mathrm{d}u} \right)^2 \\ & - \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{y-u} \right) \frac{\mathrm{d}y}{\mathrm{d}u} \\ & + \frac{y(y-1)(y-u)}{u^2(u-1)^2} \left(\alpha + \beta\frac{u}{y^2} + \gamma\frac{u-1}{(y-1)^2} + \delta\frac{u(u-1)}{(y-u)^2} \right) , \end{aligned}$$

pour les paramètres $\alpha = \dfrac{(\theta_\infty - 1)^2}{2}$, $\beta = -\dfrac{\theta_2^2}{2}$, $\gamma = \dfrac{\theta_3^2}{2}$ et $\delta = \dfrac{1-\theta_1^2}{2}$.

Le lien entre ces systèmes hamiltoniens et les déformations isomonodromiques est donné par le résultat fondamental suivant, du aux travaux de Garnier et Malmquist.

Théorème I1.9 (Garnier, Malmquist). *Pour $i \in [n-2]$, notons q_i (resp. t_i) les fonctions algébriques r_i (resp. les points $a_i \in \mathbb{P}^1$) du paragraphe précédent. Alors il existe $n-2$ fonctions algébriques p_i des variables $(t_1, \ldots, t_{n-2})$ telles que $(p_i, q_i)_i$ soit solution d'un système de Garnier $(\mathcal{G}_{n-2})$:*

$$\begin{cases} \partial_{t_k} \mathbf{p}_i & = -\partial_{\mathbf{q}_i} H_k \quad i,k \in [n] \\ \partial_{t_k} \mathbf{q}_i & = \partial_{\mathbf{p}_i} H_k \quad\quad i,k \in [n] \end{cases},$$

pour des paramètres θ_i égaux aux valeurs propres des résidus A_i.

Les solutions algébriques de systèmes de Garnier font l'objet d'une attention particulière [7, 10, 11, 27, 34] et ont même été totalement classifiées dans le cas $n = 1$ [42]. Un résultat du à Okamoto [38] (voir aussi le théorème I1.7) affirme alors que ce système vérifie la propriété de Painlevé, qui prescrit la position des singularités "compliquées" de ses solutions. Plus précisément, les seules singularités possibles pour une solution $(p_i, q_i)_{i\in[n]}$ du système (E6) en dehors des zones critiques $t_i = t_j$ (pour $i, j \in [n+2]$ tels que $i \neq j$) doivent être des pôles. Autrement dit, les seules singularités mobiles (*i.e* dépendant du choix de la solution) de l'équation sont des pôles.

I1.3 Variété des caractères d'une surface épointée

Soit Γ un groupe de type fini et soit $\mathbb{A}$ un anneau intègre; on peut alors considérer l'ensemble des représentations de Γ dans $SL_d(\mathbb{A})$ (pour $d \geq 1$):

$$\mathrm{Rep}_d(\Gamma, \mathbb{A}) := \mathrm{Hom}(\Gamma, SL_d(\mathbb{A})).$$

Considérons une présentation du groupe :

$$\Gamma = \langle a_1, \ldots, a_k \,|\, (R_n(a_1, \ldots, a_k))_{n\geq 0} \rangle \quad ;$$

alors $\mathrm{Rep}_d(\Gamma, \mathbb{A})$ s'injecte dans $SL_d(\mathbb{A})^k$ comme l'ensemble suivant :

$$\{(A_1, \ldots, A_k) \in SL_d^k \,|\, \forall n \geq 0,\, R_n(A_1, \ldots, A_k) = I_d\}.$$

Comme les équations $R_n(A_1, \ldots, A_k) = I_d$ sont polynomiales à coefficients entiers, l'ensemble $\mathrm{Rep}_d(\Gamma, \mathbb{A})$ est une sous–variété de $SL_d(\mathbb{A})^k$. En particulier, il est naturellement muni de deux topologies : celle induite par la topologie produit sur $SL_d(\mathbb{C})^k$ et la topologie de Zariski.

Une conséquence de cette description en termes de variétés algébriques est que l'on peut considérer le quotient catégorique (au sens de la théorie géométrique des invariants) :

$$\mathrm{Rep}_d(\Gamma, \mathbb{A})//SL_d(\mathbb{A})$$

de $\mathrm{Rep}_d(\Gamma, \mathbb{A})$ sous l'action diagonale de $SL_d(\mathbb{A})$ par conjugaison simultanée. Il s'agit par définition de la variété algébrique :

$$\mathrm{Spec}\left(\mathbb{C}[\mathrm{Rep}_d(\Gamma, \mathbb{A})]^{SL_d(\mathbb{A})}\right),$$

où $\mathbb{C}[\mathrm{Rep}(\Gamma, \mathbb{A})]^{SL_2(\mathbb{A})}$ est l'anneau des fonctions polynomiales sur $\mathrm{Rep}(\Gamma, \mathbb{A})$ invariantes sous l'action de $SL_d(\mathbb{A})$; on peut montrer qu'il s'agit dans le cas complexe du plus petit quotient séparé (au sens de la topologie induite) du quotient topologique $\mathrm{Rep}_d(\Gamma, \mathbb{C})/SL_d(\mathbb{C})$.

Définition I1.10. *Soit $\overline{\Sigma}$ une surface fermée compacte de genre g et soient $p_1, \ldots, p_b$ b points distincts de cette dernière. Alors le groupe fondamental Γ de la surface (non compacte) $\Sigma := \overline{\Sigma} \setminus \{p_1, \ldots, p_b\}$ est de type fini et ne dépend que du couple (g, b). La variété algébrique suivante*

$$\mathrm{Char}_d(g, b) := \mathrm{Rep}_d(\Gamma, \mathbb{C})//SL_d(\mathbb{C})$$

est alors appelée variété des caractères *de la surface Σ.*

Remarque I1.11. *Remarquons que l'on peut donner une présentation simple du groupe fondamental de la surface Σ : si $b \neq 0$, il s'agit d'un groupe libre et dans le cas contraire il est isomorphe à $\langle a_1, b_1, \ldots, a_g, b_g \,|\, [a_1, b_1] \cdots [a_g, b_g] \rangle$.*

Deux variétés de caractères vont être amenées à jouer un rôle majeur dans les travaux présentés ici : celles des sphères à quatre et cinq trous dans $SL_2(\mathbb{C})$, que nous décrivons donc brièvement ici. Dans le but d'alléger les notations dans la suite de ce texte, on pose

$$\mathrm{Char}(g, b) := \mathrm{Char}_2(g, b)\ .$$

I1.3.1 Variété des caractères de la sphère à quatre trous

On s'intéresse à la sphère de Riemann $\mathbb{S}^2$ privée de quatre points, que nous noterons $\mathbb{S}^2_4$. Si on fixe un point de base $z_0 \in \mathbb{S}^2_4$, le groupe fondamental $\pi_1(\mathbb{S}^2_4, z_0)$ est isomorphe au groupe libre $\mathbf{F}_3$ engendré, par exemple, par les trois lacets élémentaires d_1, d_2 et d_3 de la figure 1.

On souhaite étudier la variété des représentations du groupe fondamental de la sphère $\mathbb{S}^2_4$ dans $\mathbb{C}$, soit :

$$\mathrm{Rep}_2(\mathbf{F}_3, \mathbb{C}) := \mathrm{Hom}(\mathbf{F}_3, SL_2(\mathbb{C})).$$

Comme le groupe $\pi_1(\mathbb{S}^2_4, z_0)$ est un groupe libre à trois générateurs, un élément $\rho \in \mathrm{Rep}_2(\mathbf{F}_3, \mathbb{C})$ est totalement déterminé par les trois matrices $\rho(d_1), \rho(d_2)$ et $\rho(d_3)$. De ce fait on déduit

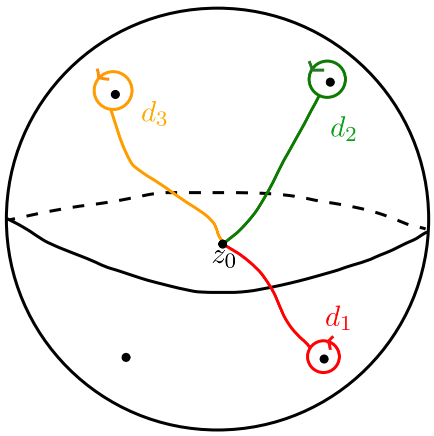

Figure 1: Groupe fondamental de la sphère épointée $\mathbb{S}^2_4$.

une bijection :

$$\mathrm{Rep}_2(\mathbf{F}_3, \mathbb{C}) \cong SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}).$$

On souhaite ici décrire la variété des caractères de $\mathbb{S}^2_4$, à savoir :

$$\mathrm{Char}(0,4) = \mathrm{Rep}_2(\mathbf{F}_3, \mathbb{C})//SL_2(\mathbb{C}).$$

Il est connu [5] que ce quotient catégorique est isomorphe à l'image de l'application suivante :

$$\begin{aligned} \chi \ : \mathrm{Rep}_2(\mathbf{F}_3, \mathbb{C}) &\to \mathbb{C}^7 \\ \rho &\mapsto (a, b, c, d, x, y, z) \end{aligned}$$

où :

$$\begin{gathered} a := \mathrm{Tr}(\rho(d_1)), \quad b := \mathrm{Tr}(\rho(d_2)), \quad c := \mathrm{Tr}(\rho(d_3)), \quad d := \mathrm{Tr}(\rho(d_1 d_2 d_3)) \\ x := \mathrm{Tr}(\rho(d_1 d_2)), \quad y := \mathrm{Tr}(\rho(d_2 d_3)), \quad z := -\mathrm{Tr}(\rho(d_1 d_3)). \end{gathered}$$

Plus précisement, on peut montrer en suivant les travaux de Benedetto et Goldman [5] que la variété $\mathrm{Char}(0,4)$ se réalise dans $\mathbb{C}^7$ comme la quartique :

$$x^2 + y^2 + z^2 = xyz + Ax + By + Cz + D \tag{E7}$$

où :

$$\begin{aligned} A \ &:= ab + cd, \quad B = bc + ad, \quad C = -(ac + bd) \\ D \ &:= 4 - a^2 - b^2 - c^2 - d^2 - abcd. \end{aligned}$$

Remarque I1.12. *Si on fixe A, B, C et D dans l'équation E7, on obtient une surface cubique $\mathcal{S}_{(A,B,C,D)}$ de $\mathbb{C}^3$. On remarque alors que $\mathcal{S}_{(0,0,0,4)} = \mathcal{S}_C$ est la cubique de Cayley. Il s'agit de la seule surface de type $(\mathcal{S}_{(A,B,C,D)})$ possédant 4 points singuliers.*

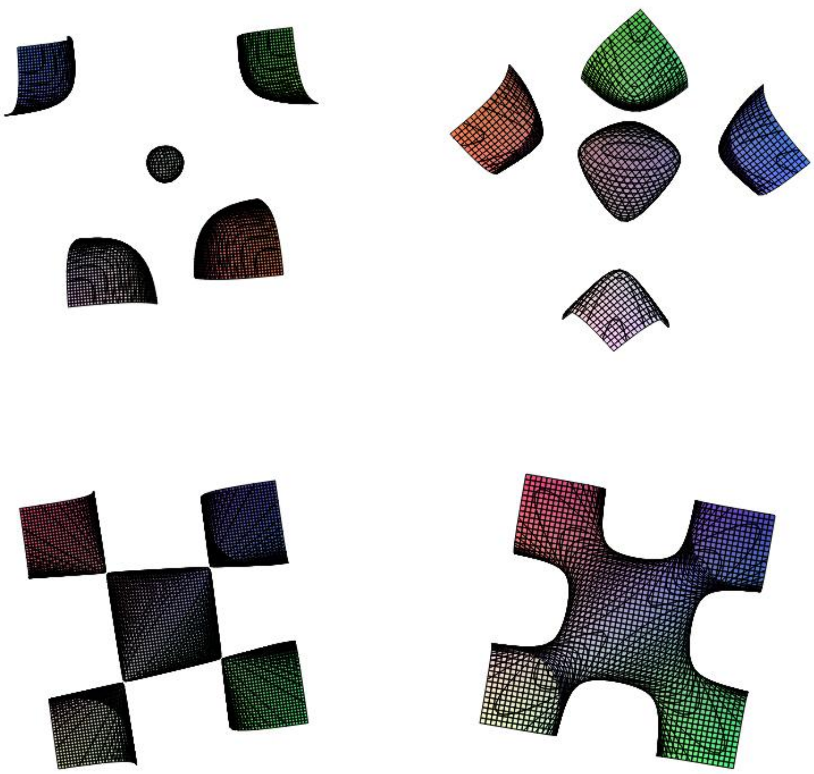

Figure 2: Surfaces $\mathcal{S}_{(0,0,0,D)}$ pour $D = \frac{1}{2}$, $D = 3$, $D = 4$ et $D = 6$ (parties réelles).

I1.3.2 Variété des caractères de la sphère à cinq trous

Dans le cas de la sphère à cinq trous, la classe d'une représentation

$$\rho \ : \mathbf{F}_4 = \langle d_1, \ldots, d_4 \,|\, \emptyset \rangle \to SL_2(\mathbb{C})$$

dans la variété de caractères $\mathrm{Char}(0,5)$ est de la même façon déterminée par les quinze fonctions coordonnées suivantes :

$$\begin{aligned}
&t_1 := \mathrm{Tr}(\rho(d_1)),\ t_2 := \mathrm{Tr}(\rho(d_2)),\ t_3 := \mathrm{Tr}(\rho(d_3)),\\
&t_4 := \mathrm{Tr}(\rho(d_4)),\ t_5 := \mathrm{Tr}(\rho(d_1 d_2 d_3 d_4)),\\
&r_1 := \mathrm{Tr}(\rho(d_1 d_2)),\ r_2 := \mathrm{Tr}(\rho(d_1 d_3)),\ r_3 := \mathrm{Tr}(\rho(d_1 d_4)),\\
&r_4 := \mathrm{Tr}(\rho(d_2 d_3)),\ r_5 := \mathrm{Tr}(\rho(d_2 d_4)),\ r_6 := \mathrm{Tr}(\rho(d_3 d_4)),\\
&r_7 := \mathrm{Tr}(\rho(d_1 d_2 d_3)),\ r_8 := \mathrm{Tr}(\rho(d_1 d_2 d_4)),\ r_9 := \mathrm{Tr}(\rho(d_1 d_3 d_4)),\ r_{10} := \mathrm{Tr}(\rho(d_2 d_3 d_4)).
\end{aligned}$$

Notons que, comme dans le cas de la sphère à quatre trous, ces dernières ne sont pas algébriquement indépendantes ; des équations explicites peuvent être obtenues de façon similaire au paragraphe précédent (voir aussi les travaux de Komyo [40] sur le sujet).

I1.3.3 Correspondance de Riemann–Hilbert

Un résultat fondamental de l'étude des connexions logarithmiques plates est la correspondance de Riemann Hilbert, qui affirme que tout point de la variété de caractères correspond à une déformation isomonodromique. Ceci justifie notre démarche de chercher à classifier certaines représentations de groupes avant de tenter de construire la déformation correspondante.

Théorème I1.13 (Riemann–Hilbert)**.** *Soit $(g, b) \in \mathbb{N}^2$; alors pour tout point $[\rho] \in \mathrm{Char}(g, b)$ il existe une connexion logarithmique plate ∇ sur le fibré trivial de rang 2 au–dessus d'une surface fermée de genre g à b trous telle que $[\rho_\nabla] = [\rho]$ dans $\mathrm{Char}(g, b)$. Réciproquement, à toute telle connexion logarithmique plate on peut associer un unique point de la variété de caractères $\mathrm{Char}(g, b)$.*

Pour plus de détails, le lecteur est invité à consulter le chapitre 3 de [58], le paragraphe III.18 de [35] ainsi que les travaux de Deligne [21] sur le sujet.

I1.4 Action du groupe modulaire

Dans ce paragraphe nous nous proposons de décrire l'action du groupe modulaire d'une surface fermée sur sa variété des caractères. Le lecteur pourra trouver plus de détails sur les groupes modulaires dans la référence [29].

Soit $\overline{\Sigma}$ une surface fermée compacte de genre g et soient $p_1, \ldots, p_b$ des points deux à deux distincts de cette dernière ; on considère la surface épointée $\Sigma := \overline{\Sigma} \setminus \{p_1, \ldots, p_b\}$. On note alors $\mathrm{Homeo}^{+,\partial}(\Sigma)$ (resp. $\underline{\mathrm{Homeo}}^{+,\partial}(\Sigma)$) l'ensemble des homéomorphismes de Σ préservant son orientation et fixant son bord $\{p_1, \ldots, p_b\}$ dans son ensemble (resp. point par point).

Définition I1.14. *Soit Σ une surface fermée de genre g à b trous. On a alors une relation d'équivalence $\sim$ sur $\mathrm{Homeo}^{+,\partial}(\Sigma)$ (resp. $\underline{\mathrm{Homeo}}^{+,\partial}(\Sigma)$) définie de la façon suivante : on note $f \sim g$ si et seulement si il existe une isotopie reliant f à g sur Σ dans $\mathrm{Homeo}^{+,\partial}(\Sigma)$ (resp. $\underline{\mathrm{Homeo}}^{+,\partial}(\Sigma)$).On définit alors les deux groupes suivants, ne dépendant à isomorphisme près que du couple (g, b) :*

1. *le* groupe modulaire *de Σ :*
$$\mathrm{Mod}(g, b) := \mathrm{Homeo}^{+,\partial}(\Sigma)/\sim \quad ;$$
2. *le* groupe modulaire pur *de Σ :*
$$\mathrm{PMod}(g, b) := \underline{\mathrm{Homeo}}^{+,\partial}(\Sigma)/\sim \ .$$

Remarque I1.15. *1. Si $b \leq 1$, on a naturellement $\mathrm{PMod}(g,b) \cong \mathrm{Mod}(g,b)$. En règle générale toutefois,* PMod *est le noyau du morphisme surjectif naturel*

$$\mathrm{Mod}(g,b) \twoheadrightarrow \mathfrak{S}_b$$

donné par l'action de $\mathrm{Homeo}^{+,\partial}(\Sigma)$ sur $\{p_1,\dots,p_b\}$. En particulier, $\mathrm{PMod}(g,b)$ est un sous–groupe distingué d'indice $b!$ dans $\mathrm{Mod}(g,b)$.

2. Notons que l'action naturelle de $\mathrm{Homeo}^{+,\partial}(\Sigma)$ sur Σ n'est pas compatible avec la relation d'isotopie, et donc ne passe pas au quotient.

Exemple I1.16. [29]

1. *le groupe modulaire de la sphère $\mathbb{S}^2$ est trivial ;*
2. *le groupe modulaire du tore $\mathbb{R}^2/\mathbb{Z}^2$ est isomorphe à $SL_2(\mathbb{Z})$;*
3. *dans le cas d'une sphère à $b \geq 1$ trous, on a l'isomorphisme suivant* [6, 29] *:*

$$\mathrm{PMod}(0,b) \cong PB_{b-1}/Z(PB_{b-1})$$

où PB_{b-1} est le groupe des tresse pures à $b-1$ brins et où pour tout groupe G la notation $Z(G)$ désigne le centre du dit groupe.

I1.4.1 Description générale

Fixons une surface fermée $\overline{\Sigma}$ de genre g ainsi que b points distincts $p_1,\dots,p_b$ de cette dernière et considérons la surface épointée $\Sigma := \overline{\Sigma} \setminus \{p_1,\dots,p_b\}$. Posons également $\Gamma := \pi_1(\Sigma, p_0)$ pour un choix arbitraire de $p_0 \in \Sigma$. Nous aurons également besoin de faire usage de l'application de quotient catégorique

$$\chi : \mathrm{Rep}_d(\Gamma,\mathbb{C}) \twoheadrightarrow \mathrm{Rep}_d(\Gamma,\mathbb{C})//SL_d(\mathbb{C}) = \mathrm{Char}(g,b)\ .$$

Le groupe $\mathrm{Aut}(\Gamma)$ des automorphismes du groupe fondamental de Σ agit naturellement sur $\mathrm{Rep}_d(\Gamma,\mathbb{C})$ de la façon suivante :

$$\begin{aligned}\mathrm{Aut}(\Gamma) &\to \mathrm{Aut}(\mathrm{Rep}_d(\Gamma,\mathbb{C}))\\ \Phi &\mapsto (\rho \mapsto \rho\circ\Phi^{-1})\ .\end{aligned}$$

On obtient de fait une action par automorphismes de $\mathrm{Aut}(\Gamma)$ sur le quotient $\mathrm{Char}_d(g,b)$ et on peut remarquer que le groupe $\mathrm{Inn}(\Gamma)$ des automorphismes intérieurs de Γ agit trivialement sur la variété de caractères $\mathrm{Char}_d(g,b)$. On a donc une action du groupe des automorphismes extérieurs $\mathrm{Out}(\Gamma) := \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ sur $\mathrm{Char}_d(g,b)$. L'injection naturelle $\mathrm{Mod}(g,b) \hookrightarrow \mathrm{Out}(\Gamma)$ [29] assure alors l'existence d'un morphisme :

$$\mathrm{Mod}(g,b) \to \mathrm{Aut}(\mathrm{Char}_d(g,b))\ .$$

I1.4.2 Le cas des sphères épointées

L'action du groupe modulaire sur la variété des caractères que nous venons de décrire a été reliée à l'étude des déformations isomonodromiques dans le cas des sphères à quatre trous dans les articles séminaux de Philip Boalch [7], Boris Dubrovin et Marta Mazzocco [27] ; des résultats similaires ont depuis été obtenus par Gaël Cousin dans le cas de sphères épointées générales [16].

Supposons que nous disposions d'une déformation isomonodromique de la sphère à n trous $\mathbb{S}^2_n$ de monodromie associée $\rho : \mathbf{F}_{n-1} \to SL_2(\mathbb{C})$; on peut alors s'intéresser à l'orbite de la classe $[\rho]$ de ρ dans la variété de caractères $\mathrm{Char}(0,n)$ sous l'action de $\mathrm{Mod}(0,n)$. Cette dernière peut être visualisée de la façon suivante : à notre déformation isomonodromique on peut associer une équation de Schlesinger (Théorème I1.8) de la forme

$$\forall i \in [n], \quad \mathrm{d}A_i = -\sum_{j \neq i} \frac{\{A_i, A_j\}}{a_i - a_j} \mathrm{d}(a_i - a_j)$$

portant sur les résidus de la connexion logarithmique plate associée. L'action des automorphismes extérieurs de $\mathbf{F}_n$ sur la variété des caractères peut alors être vue comme le fait de déformer les lacets élémentaires engendrant ce dernier (vu comme groupe fondamental de la sphère épointée) en "faisant tourner les points marqués les uns autour des autres" : cela revient de fait à étudier la monodromie de cette équation. Lorsque la déformation est algébrique, ce procédé ne peut qu'échanger les branches de la solution algébrique du système de Garnier $(\mathcal{G}_n)$ associée à cette déformation isomonodromique ce qui implique que l'orbite de $[\rho]$ sous l'action de $\mathrm{Mod}(0,n)$ doit être finie.

Dans leur article [27], Dubrovin et Mazzocco ont prouvé la réciproque dans le cas des solutions algébriques de certaines équations de Painlevé VI, elle–même équivalente au système de Garnier $(\mathcal{G}_1)$. Ce résultat a été déterminant dans la classification des solutions algébriques de cette dernière équation, ouvrant la voix à une étude qui a été systématisée par Cantat et Loray [11] puis achevée par Lisovyy et Tykhyy [42] ; citons également les travaux de Boalch [8, 9], Deift, Its, Kapaev et Zhou sur ce sujet [20].

Un résultat analogue liant orbites finies sous l'action du groupe modulaire et solutions algébriques de systèmes de Garnier a été récemment obtenu par Cousin [16] dans le cas de déformations isomonodromiques logarithmiques de rang 2 sur une sphère épointée quelconque. De fait, la recherche d'orbites finies sous cette action paraît une poursuite pertinente et viable dans le cadre de notre étude.

I1.5 Quelques avancées récentes

I1.5.1 Théorème de Corlette–Simpson

Compte tenu de nos objets d'études, nous allons être amenés à nous intéresser aux représentations de groupes fondamentaux de variétés quasi–projectives dans $SL_2(\mathbb{C})$. Ces dernières ont fait l'objet d'une étude poussée par Corlette et Simpson [15] puis par Loray, Pereira et Touzet [43], qui a fait ressortir l'importance d'une sous–classe de telles représentations, par ailleurs déja très présente dans la littérature [1] : celles se factorisant à travers une courbe.

Définition I1.17. *Soit X une variété quasi–projective complexe et soit Γ son groupe fondamental. On dit qu'une représentation $\rho\ :\Gamma \to SL_2(\mathbb{C})$* se factorise à travers une courbe *si il existe une courbe projective complexe C, un diviseur Δ (resp. δ) dans X (resp. C), un morphisme algébrique $f\ : X\setminus\Delta \to C\setminus\delta$ et une représentation $\tilde{\rho}$ du groupe fondamental de $C\setminus\delta$ dans $PSL_2(\mathbb{C})$ tels que le diagramme*

$$\begin{tikzcd} \pi_1(C\setminus\delta) \arrow[dr, "\tilde{\rho}"'] & \pi_1(X\setminus\Delta) \arrow[l, "f_*"'] \arrow[d, "\mathrm{P}\circ\rho\circ m"] \\ & PSL_2(\mathbb{C}) \end{tikzcd}$$

commute, où P et m sont les morphismes de groupes naturels $P\ : SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$ et $m\ :\pi_1(X\setminus\Delta) \to \Gamma$.

Cette classe de représentations va jouer un rôle majeur dans notre étude ; en effet on peut par exemple citer le raffinement suivant par Loray, Pereira et Touzet du résultat séminal de Corlette et Simpson.

Théorème I1.18 (Corlette–Simpson [15], Loray–Pereira–Touzet [43])**.** *Soit X une variété quasi–projective complexe et soit Γ son groupe fondamental. Alors toute représentation non–rigide (*i.e *pouvant se déformer de façon analytique dans la variété des caractères) $\rho\ :\Gamma \to PSL_2(\mathbb{C})$ d'image Zariski–dense se factorise à travers une courbe.*

I1.5.2 Solutions se factorisant à travers une courbe

À la lumière du théorème I1.18, la question suivante apparaît naturelle : *quelles sont les solutions algébriques d'un système de Garnier associées à des représentations de monodromie se factorisant à travers une courbe ?*

Les connexions plates dont la monodromie se factorise à travers une courbe correspondent à des solutions de Garnier qui sont construites par la "méthode du tiré–en–arrière" développée par Doran [26], Andreev et Kitaev [2]. Karamoko Diarra a démontré dans sa thèse [24] que seuls les systèmes de Garnier $(\mathcal{G}_n)$ pour $n \leq 3$ admettent des solutions à monodromie d'image Zariski–dense obtenues par ce procédé ; ces solutions s'obtiennent alors en tirant en arrière des connexions logarithmiques plates sur le fibré trivial de rang 2 au–dessus de $\mathbb{P}^1\setminus\{0,1,\infty\}$ (aussi appelées *équations hypergéométriques*) par des revêtements ramifiés.

La monodromie $\rho\ :\mathbf{F}_3 = \langle d_1,d_2,d_3\,|\,d_1d_2d_3 = 1\rangle \to SL_2(\mathbb{C})$ d'une connexion hypergéométrique algébrique est caractérisée (en tant que point dans $\mathrm{Char}(0,3)$) par le triplet $(p_0,p_1,p_\infty) \in (\mathbb{N}\cup\{\infty\})^3$ des ordres des matrices $\rho(d_1)$, $\rho(d_2)$ et $\rho(d_3)$. Dans [24], Diarra liste les tels triplets (p_0, p_1, p_∞) ainsi que les types de ramification des revêtements donnant lieu à des déformations isomonodromiques à monodromie Zariski–dense d'une sphère épointée, tout en précisant le degré d du revêtement et l'indice n du système de Garnier $(\mathcal{G}_n)$ correspondant ; nous reproduisons cette liste ci–dessous.

(p_0, p_1, p_∞)	d	Type de ramification	n
$(2, \infty, \infty)$	2	$(2, 1+1, 1+1)$	1
	3	$(2+1, 3, 1+1+1)$	1
$(2, 3, \infty)$	4	$(2+2, 3+1, 1+1+1+1)$	2
	6	$(2+2+2, 3+3, 1+1+1+1+1+1)$	3
$(2, 4, \infty)$	4	$(2+2, 4, 1+1+1+1)$	1
$(2, 3, 7)$	10	$(2+2+2+2+2, 3+3+3+1, 7+1+1+1)$	1
	12	$(2+2+2+2+2+2, 3+3+3+3, 7+1+1+1+1+1)$	2

I1.6 Solutions algébriques de systèmes de Garnier obtenues à l'aide de quintiques planes

Dans ce paragraphe, nous exposons les résultats de la première partie de cette thèse, qui ont trait à la classification des solutions algébriques de systèmes de Garnier obtenues à l'aide de quintiques planes.

I1.6.1 Un résultat de classification

L'objet du premier chapitre de cette thèse est de classifier les représentations de groupes $\rho : \Gamma \to PSL_2(\mathbb{C})$, où Γ est le groupe fondamental du complémentaire d'une courbe de degré cinq dans $\mathbb{P}^2(\mathbb{C})$, satisfaisant les conditions suivantes :

(C1) l'image de ρ est infinie et irréductible ;

(C2) ρ ne se factorise pas à travers une courbe (voir définition I1.17).

À la lumière de résultats obtenus par Diarra [24] et Mazzocco [46], ces deux conditions paraissent former un bon point de départ pour obtenir, à travers la correspondance de Riemann–Hilbert (théorème I1.13), une nouvelle déformation isomonodromique algébrique. En particulier, celles correspondant à une monodromie d'image Zariski–dense ne satisfaisant pas **(C2)** ont été classifiées par le premier ; nous démontrons donc le théorème suivant.

Théorème A. *Soit Γ le groupe fondamental du complémentaire d'une courbe Q de degré cinq dans $\mathbb{P}^2(\mathbb{C})$ et soit $\rho : \Gamma \to PSL_2(\mathbb{C})$ telle que :*

- *l'image de ρ est infinie et irréductible ;*
- *ρ ne se factorise pas à travers une courbe.*

Alors le triplet (Γ, ρ, Q) *est (modulo conjugaison globale à l'arrivée pour* ρ*) l'un des suivants (pour un certain couple* $(u, v) \in \mathbb{C}^* \times \mathbb{C}^*$*) :*

1. $\Gamma \cong \langle a, b, c \,|\, (ab)^2(ba)^{-2} = (ac)^2(ca)^{-2} = [b, c] = 1\rangle$,

$$\rho \ : a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$$

et Q *est formée de trois droites et d'une conique tangentes à ces dernières ;*

2. $\Gamma \cong \langle a, b, c \,|\, [a, b] = [a, c^{-1}bc] = 1,\ (bc)^2 = (cb)^2\rangle$,

$$\rho \ : a \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

et Q *est formée de trois droites concourantes et d'une conique tangente à exactement deux d'entre elles ;*

3. $\Gamma \cong \langle a, b, c \,|, [a, b] = [b, c^2] = 1,\ ca = bc\rangle$,

$$\rho \ : a \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Dans ce dernier cas Q *est toujours la réunion d'une cubique avec deux droites s'intersectant en un point de cette dernière ; la configuration exacte est détaillée dans le chapitre 1 (voir aussi la figure 3).*

La démonstration de ce résultat repose sur les travaux de classifications menés par Degtyarev [19] sur les groupes fondamentaux non abéliens de complémentaires de quintiques projectives planes. Nous donnons une description détaillée des courbes correspondants à chacun des cas cités dans le théorème A ; précisons toutefois que ce dernier ne donne qu'une condition nécessaire pour que le couple (Γ, ρ) satisfasse **(C1)** et **(C2)** ; la question de savoir dans quels cas ces conditions sont effectivement satisfaites fait l'objet des chapitres 3 et 4.

I1.6.2 Représentations de monodromie et orbites sous l'action du groupe modulaire

Dans le second chapitre de ce document, nous posons les bases de la méthode qui va nous permettre de construire des déformations isomonodromiques effectives correspondant aux représentations apparaissant dans le théorème A. Cette dernière, inspirée par les travaux de Hitchin [34] sur l'équation de Painlevé VI, repose sur la remarque suivante : si l'on dispose d'une connexion logarithmique plate ∇ au–dessus de $\mathbb{P}^2$ dont le lieu polaire est une courbe quintique alors, comme une droite générique intersecte une telle courbe en cinq points, la famille de connexions données par les restrictions de ∇ aux droites génériques du plan projectif nous livre une déformation isomonodromique de monodromie ayant même image que ρ_∇.

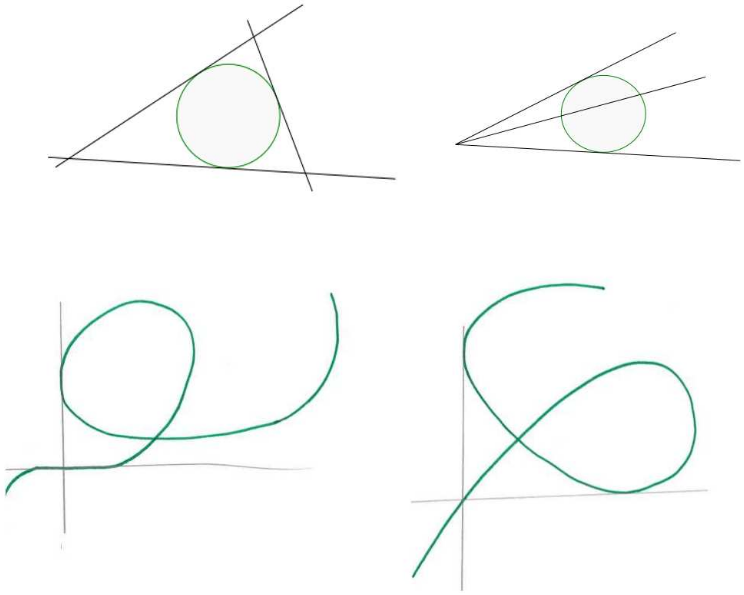

Figure 3: Courbes quintiques apparaissant dans le théorème A. En haut (de gauche à droite) les cas 1 et 2, le cas 3 en bas.

Plus précisément, soit L une droite choisie génériquement dans le plan projectif $\mathbb{P}^2(\mathbb{C})$; alors L coupe le lieu polaire de ∇ en exactement cinq points, que nous pouvons, à homographie près, supposer égaux à 0, 1, ∞ et $t_1, t_2 \in \mathbb{C}^* \setminus \{1\}$. En restreignant ∇ à la droite L, on obtient une connexion logarithmique plate au–dessus de la sphère de Riemann épointée $\mathbb{P}^1_5 := \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, t_1, t_2, \infty\}$ dont la monodromie ρ_L est donnée par le diagramme

$$\begin{tikzcd} \pi_1(\mathbb{P}^1_5) \cong \mathbf{F}_4 \arrow[r, "\tau"] \arrow[dr, "\rho_L"'] & \pi_1(\mathbb{P}^2_{\mathbb{C}} - Q) \arrow[d, "\rho_\nabla"] \\ & SL_2(\mathbb{C}) \end{tikzcd}$$

où τ est le morphisme surjectif naturel donné par le théorème de l'hyperplan de Lefschetz (voir [47], théorème 7.4). Comme la connexion ∇ est plate, il est connu [44] que ρ_L ne dépend pas (à conjugaison près) de la droite générique L et donc il existe un ouvert de Zariski U dans le dual $\widehat{\mathbb{P}^2(\mathbb{C})}$ tel que les connexions $(\nabla_L)_{L \in U}$ aient toutes la même monodromie (modulo conjugaison).

En utilisant la description explicite des courbes associées au théorème A, nous sommes ensuite en mesure de calculer les représentations de monodromie $\rho\ : \mathbf{F}_4 \to SL_2(\mathbb{C})$ associées aux couples y apparaissant ; il s'agit (dans l'ordre et à conjugaison près) de :

$$\begin{aligned}
\rho_1 \ &: d_1 \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} 0 & u^2 \\ -u^{-2} & 0 \end{pmatrix} \\
\rho_2 \ &: d_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \\
\rho_3 \ &: d_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} 0 & u^{-1} \\ -u & 0 \end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \\
\rho_4 \ &: d_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} .
\end{aligned}$$

Nous nous intéressons ensuite aux orbites sous l'action du groupe modulaire de la sphère à cinq trous des points correspondant à ces représentations dans la variété de caractères $\mathrm{Char}(0,5)$. Cette dernière est définie comme le quotient catégorique de la variété des représentations du groupe libre $\mathbf{F}_4$ dans $SL_2(\mathbb{C})$ sous l'action diagonale de $SL_2(\mathbb{C})$ par conjugaison (voir paragraphe I1.3).

Nous calculons alors de façon effective les orbites sous l'action du groupe modulaire $\mathrm{Mod}(0,5)$ sur la variété de caractères $\mathrm{Char}(0,5)$ correspondantes et prouvons le résultat suivant.

Théorème B. *On s'intéresse aux quatre familles de représentations (paramétrées par $u, v \in \mathbb{C}^*$) du groupe libre à quatre générateurs $\mathbf{F}_4 := \langle d_1, \ldots, d_4 \,|\, \emptyset \rangle$ dans $SL_2(\mathbb{C})$ $\rho_1, \ldots, \rho_4$ décrites* supra. *Alors :*

1. *les points de* $\mathrm{Char}(0,5)$ *associés à ces dernières donnent lieu à quatre orbites deux à deux distinctes de longueur quatre sous l'action du groupe modulaire pur* $\mathrm{PMod}(0,5)$ *;*

2. *les familles d'orbites sous l'action du groupe modulaire* $\mathrm{Mod}(0,5)$ *associées à ρ_1 et ρ_2 sont également distinctes ; toutefois celles associées à ρ_3 et ρ_4 sont des cas particuliers d'orbites de type ρ_2. Plus précisément, pour tous $u \in \mathbb{C}^*$ et $i = 3, 4$ il existe deux paramètres (s,t) (dépendant de u et i) tels que l'orbite de la classe de ρ_i avec paramètre u soit égale à celle de ρ_2 avec paramètres (s,t).*

Le lecteur intéressé par les orbites explicites associées à ces représentations est invité à se référer à l'annexe A.

I1.6.3 Description explicite des solutions

Les chapitres 3 et 4 sont dédiés (respectivement) à la description explicite des solutions du système de Garnier $(\mathcal{G}_2)$ associées aux familles de représentations ρ_1 et ρ_2. Dans chaque cas nous donnons une connexion explicite sur le fibré trivial de rang deux sur $\mathbb{P}^2(\mathbb{C})$ puis nous calculons sa restriction aux droites génériques et explicitons la solution algébrique de $(\mathcal{G}_2)$ associée.

Solution associée à ρ_1. Dans le chapitre 3, nous reprenons les résultats de l'article [32] concernant la solution algébrique du système $(\mathcal{G}_2)$ associée à la représentation ρ_1. Ceci signifie que nous cherchons tout d'abord à construire une connexion logarithmique plate sur le fibré trivial de rang deux au–dessus de $\mathbb{P}^2(\mathbb{C})$ dont le lieu polaire soit exactement la quintique $\mathcal{Q}$ apparaissant dans le cas 1 du théorème A d'équation :

$$xyt(x^2+y^2+t^2-2(xy+xt+yt))=0$$

composée d'une conique et de trois droites tangentes. Le groupe fondamental du complémentaire de cette courbe est isomorphe au premier groupe dans la liste du théorème A, soit :

$$\Gamma=\langle a,b,c\,|\,(ab)^2(ba)^{-2}=(ac)^2(ca)^{-2}=[b,c]=1\rangle\ .$$

On a un revêtement double ramifié naturel $\pi\ :\mathbb{P}^1\times\mathbb{P}^1\xrightarrow{2:1}\mathbb{P}^2$ qui tire la quintique $\mathcal{Q}$ en arrière sur l'ensemble $D\subset\mathbb{P}^1\times\mathbb{P}^1$ composé des six droites $u_0,u_1=0,1,\infty$ (pour (u_0,u_1) des coordonnées affines sur $\mathbb{P}^1\times\mathbb{P}^1$) et de la diagonale Δ tout en ramifiant uniquement au–dessus de cette dernière. Comme la représentation ρ_1 est diédrale donc virtuellement abélienne d'indice deux, une idée naturelle pour construire notre connexion est de définir une famille de connexions logarithmiques de rang 1 à monodromie abélienne sur $\mathbb{P}^1\times\mathbb{P}^1$ et de la pousser en avant via π. On obtient alors le résultat suivant.

Théorème C. *Il existe une famille* explicite *à deux paramètres $\nabla_{\lambda_0,\lambda_1}$ de connexions logarithmiques plates sur le fibré trivial de rang deux $\mathbb{C}^2\times\mathbb{P}^2\to\mathbb{P}^2$ ayant les propriétés suivantes :*

(i) le lieu polaire de $\nabla_{\lambda_0,\lambda_1}$ est exactement la courbe quintique $\mathcal{Q}\subset\mathbb{P}^2$ d'équation :

$$xyt(x^2+y^2+t^2-2(xy+xt+yt))=0\ \ ;$$

(ii) la monodromie de $\nabla_{\lambda_0,\lambda_1}$ est conjuguée à ρ_1 avec paramètres $u=-e^{-i\pi\lambda_0}$ et $v=e^{-i\pi\lambda_1}$.

La connexion $\nabla_{\lambda_0,\lambda_1}$ est donnée dans la carte affine $\{t=1\}\subset\mathbb{P}^2$ par :

$$\nabla_{\lambda_0,\lambda_1}=\mathrm{d}-\frac{1}{2(x^2+y^2+1-2(xy+x+y))}(\lambda_0A_0+\lambda_1A_1+A_2)\ ,$$

où

$$A_0\ :=\begin{pmatrix}2(x-1)y\mathrm{d}x+(x^2+x(y-2)-y+1)x\frac{\mathrm{d}y}{y} & 2(2x-y+2)y\mathrm{d}x+(2x^2+y(x-y+3)-2)x\frac{\mathrm{d}y}{y}\\ -2y^2\mathrm{d}x+(x+y-1)x^2\frac{\mathrm{d}y}{y} & -2(x-1)y\mathrm{d}x-(x^2+x(y-2)-y+1)x\frac{\mathrm{d}y}{y}\end{pmatrix}$$

$$A_1\ :=\begin{pmatrix}(x^2+(x-1)(y-1))y\frac{\mathrm{d}x}{x}+2(x-1)x\mathrm{d}y & (x^2+y(x-y+3)-2)y\frac{\mathrm{d}x}{x}+2(2x-y+2)x\mathrm{d}y\\ -(x+y-1)y^2\frac{\mathrm{d}x}{x}-2x^2\mathrm{d}y & -(x^2+(x-1)(y-1))y\frac{\mathrm{d}x}{x}-2(x-1)x\mathrm{d}y\end{pmatrix}$$

$$A_2\ :=\begin{pmatrix}-(x+y+1)y\mathrm{d}x-(x^2-x(y+2)-y+1)x\frac{\mathrm{d}y}{y} & -2(x-y+3)y\mathrm{d}x-(x^2-2y(x+1)+1)x\frac{\mathrm{d}y}{y}\\ 0 & (x+y+1)y\mathrm{d}x+(x^2-x(y+2)-y+1)x\frac{\mathrm{d}y}{y}\end{pmatrix}\ .$$

Partant de cela, nous sommes en mesure de donner un paramétrage rationnel explicite des solutions algébriques de $(\mathcal{G}_2)$ associées à la restriction aux droites génériques de cette famille de connexion. Nous montrons ensuite que cette famille de connexions vérifie génériquement la condition **(C2)** ; plus précisément, on prouve le résultat suivant.

Théorème D. *La représentation de monodromie des connexions $\nabla_{\lambda_0,\lambda_1}$ se factorise à travers une courbe si et seulement si il existe $(p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ tel que $p\lambda_0 + q\lambda_1 = 0$.*

Enfin, faisons la remarque suivante : si ∇ est une $\mathfrak{sl}_2(\mathbb{C})$–connexion logarithmique plate sur le fibré trivial, il existe trois 1–formes méromorphes α_0, α_1 et α_2 telles que :

$$\nabla = \mathrm{d} + \Omega \text{ , où } \Omega := \begin{pmatrix} \alpha_1 & \alpha_0 \\ -\alpha_2 & -\alpha_1 \end{pmatrix} \text{ est telle que } \mathrm{d}\Omega = \Omega \wedge \Omega \, .$$

Ici, le feuilletage défini par α_2 est en fait un feuilletage de degré deux dont le lieu invariant contient la quintique $\mathcal{Q}$. Nous montrons alors qu'il est conjugué à un feuilletage de type Lotka-Volterra sur $\mathbb{C}^3$ [48, 49]; en l'occurrence, si on se donne trois paramètres complexes (A, B, C), le feuilletage de codimension un dans $\mathbb{P}^2$ associé à la 1–forme :

$$\omega_0 := (yV_t - tV_y)\mathrm{d}x + (tV_x - xV_t)\mathrm{d}y + (xV_y - yV_x)\mathrm{d}t \ ,$$

où:

$$V_x := x(Cy + t), \qquad V_y := y(At + x) \qquad \text{and} \qquad Vt := t(Bx + y) \ .$$

Remarquons que nous avons fait ici le choix (arbitraire) d'étudier le feuilletage associé à α_2 ; la même étude peut être menée pour celui associé à α_0.

Théorème E. *Le feuilletage associé à la 1–forme méromorphe α_2 est égal au feuilletage sur $\mathbb{P}^2$ associé au système de Lotka–Volterra de paramètres*

$$(A, B, C) = \left(\frac{\lambda_1}{\lambda_0}, \frac{-\lambda_0}{\lambda_0 + \lambda_1}, \frac{-(\lambda_0 + \lambda_1)}{\lambda_1} \right) \quad .$$

Réciproquement, tout feuilletage de degré deux sur $\mathbb{P}^2$ dont le lieu invariant contient la quintique $\mathcal{Q}$ est de cette forme.

Nous démontrons enfin que ceci fournit un exemple de famille de feuilletages transversalement projectifs intégrables au sens de Liouville [48] admettant des courbes algébriques invariantes de degré arbitrairement grand (voir aussi [41]).

Solutions associée à ρ_2. De la même façon, nous cherchons à construire une famille de connexions logarithmiques plates sur le fibré trivial de rang deux au–dessus de $\mathbb{P}^2(\mathbb{C})$ dont le lieu polaire soit exactement la quintique $\mathcal{Q}'$ donnée par :

$$y(y - t)t(x^2 - yt) = 0 \ ;$$

le groupe fondamental du complémentaire de cette dernière est isomorphe au deuxième groupe apparaissant dans le théorème A, soit :

$$\langle a,b,c \,|\, [a,b]=[a,c^{-1}bc]=1,\ (bc)^2=(cb)^2\rangle\ .$$

Théorème F. *Il existe une famille* explicite *à deux paramètres $\nabla_{\lambda_0,\lambda_1}$ de connexions logarithmiques plates sur le fibré trivial de rang deux $\mathbb{C}^2\times\mathbb{P}^2\to\mathbb{P}^2$ ayant les propriétés suivantes :*

(i) le lieu polaire de $\nabla_{\lambda_0,\lambda_1}$ est exactement la courbe quintique $\mathcal{Q}\subset\mathbb{P}^2$ d'équation :

$$y(y-t)t(x^2-yt)=0\quad ;$$

(ii) la monodromie de $\nabla_{\lambda_0,\lambda_1}$ est conjuguée à ρ_2 avec paramètres $u=e^{i\pi\lambda_0}$ et $v=e^{i\pi\lambda_1}$.

La connexion $\nabla_{\lambda_0,\lambda_1}$ est donnée dans la carte affine $\{t=1\}\subset\mathbb{P}^2$ par :

$$\nabla_{\lambda_0,\lambda_1}=\mathrm{d}-\frac{1}{y(y-1)(x^2-y)}\Omega_{\lambda_0,\lambda_1}\,,$$

où

$$\Omega_{\lambda_0,\lambda_1}:=\begin{pmatrix} -\frac{(y-1)(x^2-y)}{4y}\mathrm{d}y & -\frac{2\lambda_0 y(y-1)\mathrm{d}x+(\lambda_0x(1-y)+\lambda_1(x^2-y))\mathrm{d}y}{2}y \\ -\frac{2\lambda_0 y(y-1)\mathrm{d}x+(\lambda_0x(1-y)+\lambda_1(x^2-y))\mathrm{d}y}{2} & \frac{(y-1)(x^2-y)}{4y}\mathrm{d}y \end{pmatrix}.$$

De plus, la monodromie d'une telle connexion ne factorise pas par une courbe pour λ_0,λ_1 génériques.

La construction que nous proposons ici est sensiblement identique à celle entreprise précédemment ; il s'agit avant tout de pousser en avant une famille de connexions "élémentaires" via un revêtement ramifié. Comme dans le chapitre précédent, nous donnons un paramétrage rationnel des solutions algébriques du système de Garnier associé.

I2 Convolution intermédiaire de Katz

Dans ce paragraphe, nous décrivons l'opération de convolution intermédiaire (en anglais dans la littérature *middle convolution*) des représentations d'un groupe libre par l'action du groupe de tresses introduite par Katz [37] ; nous renvoyons le lecteur aux références [22, 23, 56, 57] pour plus de détails. Avant de décrire ce procédé, nous précisons ce que nous entendons dans ce texte par *opérateur de convolution intermédiaire*. Pour un couple (g,b) d'entiers naturels, posons :

$$\mathrm{Char}_*(g,b)\ :=\bigcup_{d\in\mathbb{N}^*}\mathrm{Char}_d(g,b)\quad ;$$

ensemble sur lequel nous pouvons définir une action du groupe modulaire $\mathrm{Mod}(g,b)$ d'après le paragraphe I1.4.

Définition I2.1. *Soit S (resp. S') une surface fermée de genre g (resp. g') privée de b (resp. b') points et soit G un groupe agissant sur les variétés de caractères* $\mathrm{Char}_d(g,b)$ *et* $\mathrm{Char}_d(g',b')$ *pour tout $d \geq 1$. On appelle* convolution intermédiaire *entre S et S' relativement au groupe G toute application G–équivariante*

$$\mathfrak{C} \ : \mathrm{Char}_*(g,b) \to \mathrm{Char}_*(g',b') \ .$$

I2.1 Procédé général

I2.1.1 Action du groupe de tresses sur un produit cartésien

Pour un entier $r \geq 1$, on définit le groupe de tresses d'Artin à r brins par la présentation suivante :

$$B_r \ := \langle \sigma_1, \ldots, \sigma_{r-1} \,|\, [\sigma_i, \sigma_j] = 1 \text{ si } |i-j| > 1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle \ .$$

Si G est un groupe quelconque, on a une action à droite naturelle de B_r sur l'ensemble produit G^r définie de la façon suivante ; si $\underline{g} = (g_1, \ldots, g_r) \in G^r$ et $i \in [r]$ on pose :

$$\underline{g}^{\sigma_i} \ := (g_1, \ldots, g_{i-1}, \underbrace{g_{i+1}}_{i}, \underbrace{g_{i+1}^{-1} g_i g_{i+1}}_{i+1}, \ldots, g_r) \ .$$

Un sous–groupe intéressant de B_r est le groupe de tresses pures PB_r, défini comme le noyau du morphisme naturel :

$$\begin{aligned} B_r &\to \mathfrak{S}_r \\ \sigma_i &\mapsto (i \quad i+1) \ ; \end{aligned}$$

ce sous–groupe est engendré par les tresses suivantes, pour $1 \leq i \leq j < r$

$$\sigma_{i,j} \ := (\sigma_j \ldots \sigma_{i+1}) \sigma_i^2 (\sigma_j \ldots \sigma_{i+1})^{-1} \ .$$

Posons à présent $F_i \ := \sigma_{i,r-1}$ pour $i \in [r-1]$; alors pour tout $\underline{g} \in G^r$ on a :

$$\underline{g}^{F_i} = (g_1, \ldots, g_{i-1}, \, g_r g_i g_r^{-1}, \, [g_i^{-1}, g_r^{-1}] g_{i+1} [g_r^{-1}, g_i^{-1}], \ldots, [g_i^{-1}, g_r^{-1}] g_{r-1} [g_r^{-1}, g_i^{-1}], \, g_i g_r g_i^{-1}) \ , \tag{E8}$$

où $[g,h] \ := ghg^{-1}h^{-1}$.

I2.1.2 Action par tresses sur les extensions affines de représentations d'un groupe libre

Dans tout ce paragraphe, on se donne une représentation $\rho \ : \mathbf{F}_{r-1} \to GL_d(\mathbb{C})$ du groupe libre à $r-1$ générateurs $\mathbf{F}_{r-1} \ := \langle d_1, \ldots, d_r \,|\, d_1 \ldots d_r = 1 \rangle$; notons que l'on peut assimiler ρ à un unique élément $\underline{g} \in GL_d(\mathbb{C})^r$ tel que $g_1 \ldots g_r = I_d$, en posant $g_i \ := \rho(d_i)$. Nous supposons également que ρ satisfait les conditions suivantes (*cf.* I2.2 pour une construction explicite de telles représentations dans notre cadre d'étude) :

(CIK1) la matrice g_r commute à tous les g_i, $i \in [r-1]$;

(CIK2) 1 n'est pas valeur propre de la matrice g_r.

Espace des extensions affines. Rappelons brièvement que si d est un entier supérieur ou égal à un et k un corps on peut définir le *groupe affine* d–dimensionnel sur k comme le produit semi–direct

$$GA_d(k) \ := k^d \rtimes GL_d(k)$$

où le produit de deux éléments $(u,g),(v,h) \in GA_d(k)$ est défini comme :

$$(u,g) \cdot (v,h) \ := (u + g \cdot v, gh) \ .$$

La deuxième composante d'un couple $(u,g) \in GA_d(k)$ est appelée sa *partie linéaire* ; si elle est égale à la matrice identité I_d, on dira que l'application affine correspondante est une *translation*.

L'action par tresses (au sens du paragraphe précédent) de B_r sur $GL_d(\mathbb{C})^r$ peut donc être vue comme une action sur $\mathrm{Hom}(\mathbf{F}_{r-1}, GL_d(\mathbb{C}))$, qui coïncide avec l'action du groupe modulaire définie dans le paragraphe I1.4. On souhaite ici s'intéresser aux *extensions affines* de ρ, *i.e* aux extensions de ρ au groupe affine $GA_d(\mathbb{C})$. Posons :

$$U_\rho \ := \Big\{ \underline{u} \in (\mathbb{C}^d)^r \,|\, (u_1, g_1) \ldots (u_r, g_r) = (0, I_d) \text{ et } \forall i \in [r], u_i \in \mathrm{Im}(I_d - g_i) \Big\} \quad ;$$

alors pour tout $\underline{u} \in U_\rho$, l'application

$$\rho_{\underline{u}} \ : d_i \mapsto (u_i, g_i)$$

définit une représentation du groupe $\mathbf{F}_{r-1}$ dans le groupe affine $GA_d(\mathbb{C})$. De plus, on vérifie rapidement que la condition $u_i \in \mathrm{Im}(I_d - g_i)$ est équivalente au fait que (u_i, g_i) soit conjugué (par une translation) dans le groupe affine à sa partie linéaire $(0, g_i)$. On peut donc définir l'espace des extensions affines admissibles de ρ comme :

$$\mathrm{Aff}(\rho) \ := \{ \rho_{\underline{u}} \,|\, \underline{u} \in U_\rho \} \ .$$

Toutefois, remarquons que B_r n'agit pas sur l'espace $\mathrm{Aff}(\rho)$; en effet l'action par tresses ne laisse pas invariante la partie linéaire.

Structure de U_ρ et action par tresses. Soit $\underline{u} \in \mathrm{Im}(I_d - g_1) \times \cdots \times \mathrm{Im}(I_d - g_r)$; remarquons alors que :

$$\begin{aligned}\underline{u} \in U_\rho &\Leftrightarrow (u_1, g_1) \dots (u_r, g_r) = (0, I_d)\\ &\Leftrightarrow u_1 + g_1 u_2 + \dots + g_1 \dots g_{r-1} u_r = 0\\ &\Leftrightarrow \underline{u} \in \mathrm{Ker}(\varphi(\rho)) ,\end{aligned}$$

où $\varphi(\rho)$ est la matrice par blocs

$$\left(\begin{array}{c|c|c|c|c} I_d & g_1 & g_1 g_2 & \dots & g_1 \dots g_{r-1} \end{array} \right) .$$

Ainsi, U_ρ est un sous–espace vectoriel de $\mathrm{Im}(I_d - g_1) \times \cdots \times \mathrm{Im}(I_d - g_r)$ de codimension d ; autrement dit on a l'égalité

$$\dim(U_\rho) = \sum_{i=1}^{r} \mathrm{rang}(I_d - g_i) - d . \tag{E9}$$

Comme $\mathrm{rang}(I_d - g_r) = d$ de par la condition **(CIK2)**, on peut simplifier cette expression en

$$\dim(U_\rho) = \sum_{i=1}^{r-1} \mathrm{rang}(I_d - g_i) . \tag{E10}$$

Considérons le sous–groupe suivant du groupe de tresses B_r :

$$G_\rho := \{\sigma \in B_r \,|\, \underline{g}^\sigma = \underline{g}\} .$$

Ce groupe est non trivial ; en effet la condition **(CIK1)** combinée à l'équation (E8) nous livre que pour tout $i \in [r-1]$ la tresse pure

$$F_i = (\sigma_{r-1} \dots \sigma_{i+1}) \sigma_i^2 (\sigma_{r-1} \dots \sigma_{i+1})^{-1}$$

appartient à G_ρ.

On peut alors vérifier que G_ρ agit (à droite) sur $\mathrm{Aff}(\rho)$. En effet, la partie linéaire est préservée par construction et un calcul rapide [57] montre que les deux conditions définissant U_ρ sont bien préservées. En particulier, ceci signifie que si $\underline{u} \in U_\rho$, alors

$$\forall \sigma \in G_\rho,\ \exists \underline{v} \in U_\rho,\ \rho_{\underline{u}}^\sigma = \rho_{\underline{v}} \ ;$$

ce qui signifie que le groupe G_ρ agit à droite sur l'espace vectoriel U_ρ.

Calculs explicites. Fixons $\underline{u} \in U_\rho$ et $i \in [r-1]$; alors on peut calculer explicitement l'image de la représentation $\rho_{\underline{u}}$ sous l'action de la tresse σ_i, en l'occurrence :

$$\begin{aligned}\rho_{\underline{u}}^{\sigma_i} &= ((u_1, g_1), \ldots, (u_r, g_r))^{\sigma_i} \\ &= ((u_1, g_1), \ldots, (u_{i-1}, g_{i-1}, \underbrace{(u_{i+1}, g_{i+1})}_{i}, \underbrace{(g_{i+1}^{-1}u_i + g_{i+1}^{-1}(g_i - I_d)u_{i+1},\, g_{i+1}^{-1}g_ig_{i+1})}_{i+1}, \ldots (u_r, g_r)) \,.\end{aligned}$$

Ainsi, $\rho_{\underline{u}}$ est envoyée sur l'extension affine de $\underline{g}^{\sigma_i}$ associée au vecteur $M_{\sigma_i}\underline{u}$, où M_{σ_i} est la matrice par blocs :

$$M_{\sigma_i} \;:=\; \left(\begin{array}{c|c|c} I_{i-1} & & \\ \hline & \begin{matrix} 0 & g_{i+1}^{-1} \\ g_{i+1} & g_{i+1}^{-1}(g_i - I_d) \end{matrix} & \\ \hline & & I_{r-i-1} \end{array}\right)$$

où le bloc spécial est situé au niveau des colonnes/lignes i et $i+1$. Partant de cette remarque, on peut donner des formules explicites pour l'action de G_ρ sur U_ρ ; en l'occurrence si $\underline{u} \in U_\rho$ alors $\underline{u}^{F_i} = N_i\underline{u}$ où N_i est la matrice obtenue comme suit : partant de la matrice identité I_r, on remplace ses i–ème et r–ième colonnes par (dans l'ordre) :

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ g_r \\ g_i(I_d - g_r)(I_d - g_{i+1}^{-1}) \\ \vdots \\ g_i(I_d - g_r)(I_d - g_{r-1}^{-1}) \\ g_i(I_d - g_r) \end{pmatrix} \quad \text{et} \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g_r(g_i^{-1} - I_d) \\ g_r(g_i - I_d)(I_d - g_{i+1}^{-1}) \\ \vdots \\ g_r(g_i - I_d)(I_d - g_{r-1}^{-1}) \\ I_d - g_r + g_ig_r \end{pmatrix}$$

où à chaque fois le premier coefficient non nul se situe ligne i.

I2.1.3 Opérateur de convolution intermédiaire

Extensions triviales. Nous avons vu dans le paragraphe précédent que les extensions affines de ρ conjuguées en chaque générateur à leur partie linéaire sont stables sous l'action d'un sous–groupe particulier du groupe de tresses B_r. Supposons à présent que $\rho_{\underline{u}} \in \mathrm{Aff}(\rho)$

est globalement conjuguée par une translation à sa partie linéaire, *i.e* que :

$$\exists v \in \mathbb{C}^d \;\; \forall i \in [r],\;\; (v, I_d)\cdot(u_i, g_i)\cdot(-v, I_d) = (0, g_i)\;.$$

On remarque que comme le membre de gauche de cette égalité est égal à $(u_i + (I_d - g_i)v, g_i)$ l'existence d'un tel v est équivalente au fait que $\underline{u} \in \mathrm{Im}(\psi(\rho))$, où

$$\psi(\rho) \;:=\; \begin{pmatrix} I_d - g_1 \\ \vdots \\ I_d - g_r \end{pmatrix}\;.$$

Posons donc :

$$V_\rho \;:= \{\psi(\rho)v \mid v \in \mathbb{C}^d\}\;\;;$$

et remarquons que l'on a l'inclusion $V_\rho \subset U_\rho$. On vérifie ensuite en utilisant les formules explicites données plus haut que si $i \in [r-1]$ et $v \in \mathbb{C}^d$ alors les coordonnées $(i, i+1)$ de

$$\rho^{\sigma_i}_{\psi(\rho)v} = (((I_d - g_1)v, g_1), \ldots, ((I_d - g_r)v, g_r))^{\sigma_i}$$

sont égales à (dans l'ordre) :

$$((I_d - g_{i+1})v, g_{i+1}) \text{ et } (g_{i+1}^{-1}(I_d - g_i)v + g_{i+1}^{-1}(g_i - I_d)(I_d - g_{i+1})v,\; g_{i+1}^{-1}g_i g_{i+1})\;.$$

En développant cette dernière expression, on trouve que

$$\rho^{\sigma_i}_{\psi(\rho)v} = (((I_d - h_1)v, h_1), \ldots, ((I_d - h_r)v, h_r)\;,$$

où

$$h := \underline{g}^{\sigma_i} = (g_1, \ldots, g_{i-1}, \underbrace{g_{i+1}}_{i}, \underbrace{g_{i+1}^{-1}g_i g_{i+1}}_{i+1}, \ldots, g_r)\;.$$

En particulier, ceci implique que G_ρ, en tant que stabilisateur de $\underline{g}$ dans B_r, agit trivialement sur V_ρ.

Convolution intermédiaire de Katz. D'après le paragraphe précédent, on a une action naturelle de G_ρ sur l'espace vectoriel quotient

$$E_\rho \;:= U_\rho / V_\rho\;.$$

De plus, comme la matrice $I_d - g_r$ est inversible de part la condition **(CIK2)**, la matrice $\psi(\rho)$ est de rang d. L'équation (E10) nous permet alors d'écrire :

$$d_\rho \;:= \dim(E_\rho) = \sum_{i=1}^{r-1} \mathrm{rang}(I_d - g_i) - d\;. \tag{E11}$$

Il est de plus clair d'après les formules décrivant leur action sur U_ρ que les tresses pures F_i agissent linéairement sur E_ρ. Nous sommes maintenant en mesure de définir la *convolution intermédiaire* de la représentation ρ sous l'action du groupe de tresses B_r. Notons que le lien entre ce procédé et la définition I2.1 sera détaillé dans le paragraphe suivant.

Définition I2.2. [37, 57] *Soit ρ une représentation du groupe libre à $r-1$ générateurs $\mathbf{F}_{r-1} := \langle d_1, \ldots, d_r \,|\, d_1 \ldots d_r = 1\rangle$ dans $GL_d(\mathbb{C})$ vérifiant les conditions **(CIK1)** et **(CIK2)**. Pour $i \in [r-1]$, on considère l'application linéaire $\tilde{g}_i \in GL_{d_\rho}(\mathbb{C})$ correspondant à l'action de la tresse pure F_i sur E_ρ et on pose $\tilde{g}_r := (\tilde{g}_1 \ldots \tilde{g}_{r-1})^{-1}$. On définit la* convolution intermédiaire de Katz *de la représentation ρ comme la représentation*

$$\begin{aligned} \mathfrak{K}(\rho) : \mathbf{F}_{r-1} &\to GL_{d_\rho}(\mathbb{C}) \\ d_i &\mapsto \tilde{g}_i \,. \end{aligned}$$

*De plus, $\mathfrak{K}(\rho)$ vérifie les conditions **(CIK1)** et **(CIK2)**.*

I2.2 Application à l'étude des orbites sous l'action de $\mathrm{Mod}(0, n)$

I2.2.1 Équivariance de la convolution intermédiaire sous l'action du groupe modulaire

La convolution intermédiaire de Katz fournit un outil intéressant dans l'étude des orbites finies sous l'action du groupe modulaire $\mathrm{Mod}(0, n)$, que nous nous proposons de décrire dans ce paragraphe. Supposons que nous disposions d'une représentation $\rho : \mathbf{F}_{n-1} \to SL_d(\mathbb{C})$ du groupe libre à $n-1$ générateurs $\mathbf{F}_{n-1} = \langle d_1, \ldots, d_n \,|\, d_1 \ldots d_n = 1\rangle$; on lui associe alors le n–uplet des matrices $h_i := \rho(d_i) \in SL_d(\mathbb{C})$. En fixant n nombres complexes non nuls θ_i tels que $\theta_1 \ldots \theta_n \neq 1$ on peut alors poser :

$$\forall i \in [n],\ g_i := \frac{1}{\theta_i} h_i \text{ et } g_{n+1} := \left(\prod_{i=1}^{n} \theta_i\right) I_d \,.$$

Le n+1–uplet $\underline{g} = (g_1, \ldots, g_{n+1})$ détermine alors une unique représentation $\tilde{\rho} : \mathbf{F}_{n+1} \to GL_d(\mathbb{C})$ vérifiant les conditions **(CIK1)** et **(CIK2)**. En utilisant les formules explicites données précédemment pour le calcul de $\underline{\tilde{g}} := \mathfrak{K}(\tilde{\rho})$, on remarque que g_{n+1} doit être de la forme λI_{d_ρ}. Cette construction permet de définir pour chaque choix de $\underline{\theta}$ une application sur la variété de caractères $\mathrm{Char}_d(0, n)$ en vérifiant que la quantité :

$$\mathfrak{K}_{\underline{\theta}}([\rho]) := \left[\left(\frac{1}{\tau_1}\tilde{g}_1, \ldots, \frac{1}{\tau_{r-1}}\tilde{g}_{r-1}\,, \frac{\tau_1 \ldots \tau_{r-1}}{\tau_r \lambda}\tilde{g}_r\right)\right]$$

où $\tau_i^{d_\rho} = \det(\tilde{g}_i)$, ne dépend pas du choix du représentant ρ dans cette dernière (elle dépend par contre du choix de $\underline{\theta}$). On peut alors montrer que cette dernière constitue bien une convolution intermédiaire au sens de la définition I2.1 [22, 23].

Théorème I2.3. *L'application $\mathfrak{K}_{\underline{\theta}}$ définie ci–dessus est une convolution intermédiaire entre la sphère à n trous et elle–même relativement au groupe modulaire pur* $\mathrm{PMod}(0,n)$.

Comme $\mathrm{PMod}(0,n)$ est isomorphe au quotient du groupe des tresses pures PB_{n-1} par son centre alors cette application est $\mathrm{PMod}(0,n)$ équivariante par construction, *i.e* si $\tau \in \mathrm{PMod}(0,n)$ et $[\rho] \in \mathrm{Char}_d(0,n)$ alors :

$$\mathfrak{K}_{\underline{\theta}}(\tau \cdot [\rho]) = \tau \cdot \mathfrak{K}_{\underline{\theta}}([\rho])$$

où $\tau \cdot [\rho]$ (resp. $\tau \cdot \mathfrak{K}_{\underline{\theta}}([\rho])$) désigne l'image de $[\rho]$ (resp. $\mathfrak{K}_{\underline{\theta}}([\rho])$) sous l'action de $\mathrm{PMod}(0,n)$ sur $\mathrm{Char}_d(0,n)$ (resp. $\mathrm{Char}_{d_\rho}(0,n)$). En effet, ce processus mélange convolution intermédiaire de Katz (au sens du paragraphe précédent) et tensorisation par une représentation de rang un, qui sont bien deux opérations $\mathrm{PMod}(0,n)$ équivariantes. Une conséquence fondamentale de cette équivariance est la suivante : *si l'orbite de* $[\rho]$ *sous l'action de* $\mathrm{Mod}(0,n)$ *est finie, alors celle de* $\mathfrak{K}_{\underline{\theta}}([\rho])$ *l'est également.*

I2.2.2 Le cas $\mathrm{Mod}(0,4)$

La convolution intermédiaire de représentations a joué un rôle majeur dans la classification des solutions algébriques des équations de Painlevé VI ; comme nous l'avons souligné précédemment, ces dernières sont en correspondance avec les orbites finies sous l'action du groupe modulaire $\mathrm{Mod}(0,4)$ sur la variété de caractères $\mathrm{Char}(0,4)$. Dans son article [7], Boalch met en place le procédé suivant : partant de quatre matrices $h_1, \ldots, h_4 \in SL_3(\mathbb{C})$ engendrant un groupe fini (de réflexions) et choisies de telle sorte que h_1, h_2 et h_3 aient une valeur propre double θ_i on choisit $\theta_4 \in \mathbb{C}^*$ de telle sorte que

- $\theta_1 \ldots \theta_4 \neq 1$;
- $\mathrm{rang}(I_3 - \theta_4^{-1} h_4) = 2$.

Si $\rho \ : \mathbf{F}_3 \to SL_3(\mathbb{C})$ est la représentation associée à (h_1, h_2, h_3, h_4), on peut alors considérer sa convolution intermédiaire $\mathfrak{K}_{\underline{\theta}}([\rho])$. D'après l'équation (E11), cette dernière est une représentation de rang $1+1+1+2-3 = 2$. De plus, comme le groupe $\langle h_1, \ldots, h_4 \rangle \leq SL_3(\mathbb{C})$ est fini, l'orbite de $[\rho]$ sous l'action de $\mathrm{Mod}(0,4)$ est finie ; c'est donc également le cas de celle de $\mathfrak{K}_{\underline{\theta}}([\rho])$.

Ce procédé a été utilisé par Boalch [7] pour générer des orbites finies sous cette dernière action, ce qui lui a permis de calculer les solutions algébriques de Painlevé VI. Cette étude a été poursuivie par Filipuk [30], Dettweiler et Reiter [22, 23] qui ont étudié les symétries de l'équation de Painlevé VI obtenues par convolution intermédiaire. Plus précisément, si on part cette fois d'une solution algébrique d'une telle équation, de représentation de monodromie associée $\rho \ : \mathbf{F}_3 \to SL_2(\mathbb{C})$ ayant une orbite finie sous l'action du groupe modulaire, alors sa convolution intermédiaire via $\underline{\theta} \in (\mathbb{C}^*)^4$ choisi de façon à ce que θ_i soit valeur propre simple de $\rho(d_i)$ est également de rang deux (par l'équation E11) et donc correspond également à une solution algébrique d'une équation de Painlevé VI (pour des paramètres a priori différents). Filipuk a montré que ce type d'opération est relié aux symétries mis en évidence par Okamoto [51] dans les années 1980.

I2.3 Résultats originaux relatifs aux convolutions intermédiaires

Dans ce paragraphe, nous décrivons les résultats démontrés dans la seconde partie de cette thèse, en nous intéressant aux solutions qu'il est possible d'obtenir par convolution intermédiaire de Katz avant de décrire les prémices d'un procédé analogue en genre 1.

I2.3.1 Quelques nouvelles orbites

Le chapitre 5 de cette thèse est dédié à l'adaptation partielle d'un travail mené par Boalch [7] dans le cas de l'équation de Painlevé VI au système de Garnier ($\mathcal{G}_2$). Plus précisément, nous utilisons la convolution intermédiaire de Katz (*cf* section I2) pour obtenir des orbites finies dans la variété de caractères $\mathrm{Char}(0,5)$ sous l'action du groupe modulaire $\mathrm{Mod}(0,5)$ à partir de sous–groupes finis de $SL_3(\mathbb{C})$ (voir paragraphe I1.4).

Théorème G. *On considère le groupe libre à quatre générateurs* $\mathbf{F}_4 := \langle d_1,\ldots,d_4 \,|\, \emptyset\rangle$. *Alors, les classes dans la variété des caractères des représentations suivantes ont une orbite finie sous l'action du groupe modulaire* $\mathrm{Mod}(0,5)$ *:*

$$\rho_1 : d_1 \mapsto \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad ;$$

$$\rho_2 : d_1 \mapsto \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ \frac{9}{2} & -\frac{1}{2} \end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \quad ;$$

$$\rho_3 \ : d_1 \mapsto \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} \frac{-15+i\sqrt{3}}{6} & -\frac{2\sqrt{3}+4i}{3} \\ -i & \frac{9-i\sqrt{3}}{6} \end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} \frac{9-i\sqrt{3}}{6} & \frac{\sqrt{3}+i}{6} \\ i & \frac{3+i\sqrt{3}}{6} \end{pmatrix} \ .$$

Ces orbites sont de plus deux à deux distinctes.

Ce théorème est plus à considérer comme une motivation pour poursuivre cette étude que comme un résultat à part entière ; en effet l'étude que nous avons menée ici ne l'a pas été faite de façon systématique, et nous n'avons pas encore de description explicite (au sens des chapitres 3 et 4) des solutions algébriques d'un système de Garnier associées. Nous donnons toutefois un algorithme effectif, implémenté sous `Maple` pour le calcul de la convolution intermédiaire d'une représentation d'un groupe libre donnée, et explorons une piste liée à la confluence des pôles du système de Garnier considéré.

I2.3.2 Convolution intermédiaire elliptique virtuelle

Dans le chapitre 6 nous nous intéressons au développement d'une possible extension de la convolution intermédiaire de Katz au cas d'un tore à deux trous, principalement en adaptant le formalisme donné par Völklein [57]. Cependant, nous ne parvenons à obtenir qu'une généralisation partielle de ce procédé.

Soient x_0, p_1, p_2 et p_3 quatre points deux à deux distincts du tore complexe $\mathbb{T}^2 := \mathbb{C}/\mathbb{Z}^2$ et soit Γ_2 (resp. Γ_3) le groupe fondamental $\pi_1(\mathbb{T}^2 \backslash \{p_1, p_2\}, x_0)$ (resp. $\pi_1(\mathbb{T}^2 \backslash \{p_1, p_2, p_3\}, x_0)$). Il est connu que le groupe Γ_i $(i = 2, 3)$ admet la présentation suivante (avec les notations de la figure 4) :

$$\Gamma_2 = \langle \gamma_1, \gamma_2, \alpha, \beta \,|\, \gamma_1\gamma_2[\alpha, \beta] = 1 \rangle \quad \text{et} \quad \Gamma_3 = \langle \gamma_1, \gamma_2, \gamma_3, \alpha, \beta \,|\, \gamma_1\gamma_2\gamma_3[\alpha, \beta] = 1 \rangle \ .$$

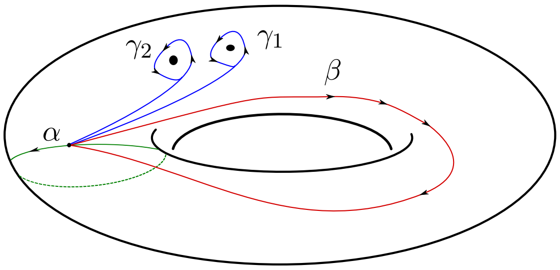

Figure 4: Générateurs de Γ_2.

Alors pour tout lacet $\delta \in \pi_1(\mathbb{T}^2 \setminus \{p_1, p_2\}, p_3)$, on peut déformer continûment les lacets γ_1, γ_2, γ_3, α et β en déplaçant p_3 le long de δ. Ceci définit un morphisme de groupes

$$\Gamma_2 \to \mathrm{Out}(\Gamma_3) \ .$$

Théorème H. *Pour tout $r \geq 2$, il existe une convolution intermédiaire (au sens de la définition I2.1) explicite entre le tore à deux trous et le tore à $2r$ trous relativement à l'action du groupe fondamental du tore à deux trous décrite ci–dessus.*

Ce résultat est plus faible que la construction de Katz dans le sens où nous n'obtenons qu'un représentation d'un sous–groupe d'indice fini du groupe initial.

Part I

Algebraic Garniers solutions obtained using plane quintic curves

Chapter 1

A classification result

The object of this chapter is to classify the group representations $\rho : \Gamma \to PSL_2(\mathbb{C})$, where Γ is the fundamental group of the complement of some quintic curve in $\mathbb{P}^2(\mathbb{C})$, that satisfy the following conditions:

(C1) the image of ρ is irreducible and infinite;

(C2) ρ does not factor through a curve (see Definition 1.1.2).

Let C be an irreducible component of some quintic curve Q.If U is a sufficiently small analytic neighbourhood of some smooth point $p \in C$ one gets:

$$\pi_1(U \setminus C \cap U) \cong \mathbb{Z} .$$

Let γ be any loop generating the above cyclic group; then $\rho(\gamma) \in PSL_2(\mathbb{C})$ must be non–trivial. Indeed, one would otherwise get a factorisation of ρ through the fundamental group of the complement of some curve in $\mathbb{P}^2$ with degree at most 4 in which case earlier work by Cousin (see Section 5.1 in [17]) proves that ρ cannot satisfy both **(C1)** and **(C2)**. In this chapter, we prove the following result.

Theorem A. *Let Γ be the fundamental group of the complement of some quintic curve Q in $\mathbb{P}^2(\mathbb{C})$ and let $\rho : \Gamma \to PSL_2(\mathbb{C})$ satisfy the following conditions:*

(C1) *the image of ρ is irreducible and infinite;*

(C2) *ρ does not factor through a curve.*

Then the triple (Γ, ρ, Q) is (up to global conjugacy for ρ and $PGL_3(\mathbb{C})$ action for Q) one of the following:

1. $\Gamma \cong \langle a, b, c \,|\, (ab)^2(ba)^{-2} = (ac)^2(ca)^{-2} = [b, c] = 1\rangle$,

$$\rho : a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \text{ , for some } u, v \in \mathbb{C}^* \text{ ,}$$

and the curve Q is composed of three lines tangent to a conic ;

2. $\Gamma \cong \langle a, b, c \,|\, [a, b] = [a, c^{-1}bc] = 1,\ (bc)^2 = (cb)^2 \rangle$,

$$\rho : a \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ , for some } u, v \in \mathbb{C}^* \text{ ,}$$

and Q is made of three concurrent lines and a conic tangent to two of the latter ;

3. $\Gamma \cong \langle a, b, c \,|, [a, b] = [b, c^2] = 1,\ ca = bc \rangle$,

$$\rho : a \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ , for some } t \in \mathbb{C}^* \text{ .}$$

Here, the curve Q is one of two special configurations of two lines and a cubic, described in Section 1.2.4.2.

*Note that the above triples do occur but do not necessarily satisfy conditions **(C1)** and **(C2)**, depending on the parameters.*

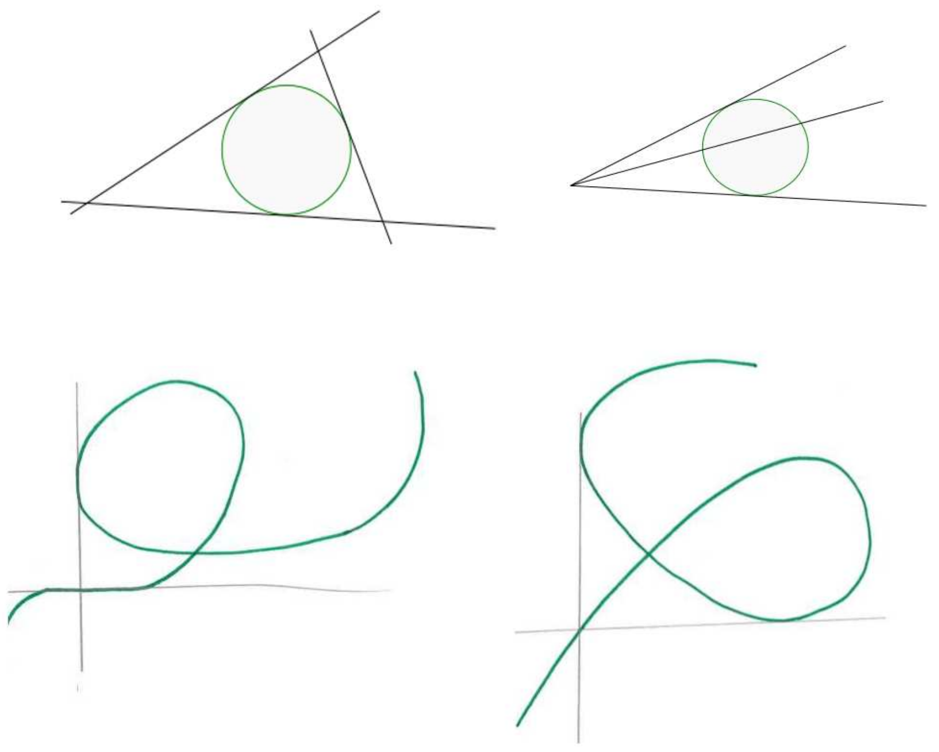

Figure 1.1: Quintic curves appearing in Theorem A. On the top row (from left to right) are case 1 and 2, case 3 appears in the bottom row.

1.1 Preliminary remarks

1.1.1 The Corlette–Simpson theorem

First, we recall a well known fact about the rank two projective special linear group $PSL_2(\mathbb{C})$; to do so start by setting $\mathrm{P} : SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$ to be the canonical projection.

Proposition 1.1.1. [14] *Let H be an infinite non Zariski–dense algebraic subgroup of $PSL_2(\mathbb{C})$; then H is contained (up to conjugacy) in the image of one of the following groups in $PSL_2(\mathbb{C})$:*

- *the group of triangular 2×2 matrices in $SL_2(\mathbb{C})$;*
- *the infinite dihedral group:*

$$\mathbf{D}_\infty := \left\{ \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{C}^* \right\} \leq SL_2(\mathbb{C})\,.$$

Representations of fundamental groups of quasi-projective varieties in $SL_2(\mathbb{C})$ have been classified mainly by Corlette and Simpson [15]. One important class of such representations is that of those factoring through a curve.

Definition 1.1.2. [15, 43] *Let Γ be the fundamental group of the complement of some curve in $\mathbb{P}^2(\mathbb{C})$. We say that a representation $\rho : \Gamma \to PSL_2(\mathbb{C})$* factors through a curve *if there exists a complex projective curve C, a divisor Δ (resp. δ) in $\mathbb{P}^2$ (resp. C), an algebraic mapping $f : \mathbb{P}^2 - \Delta \to C - \delta$ and a representation $\tilde{\rho}$ of the fundamental group of $C - \delta$ into $PSL_2(\mathbb{C})$ such that*

(i) Δ contains $\mathcal{Q}$, therefore there exists a natural group homomorphism $m : \pi_1(\mathbb{P}^2 - \Delta) \to \Gamma_2$;

(ii) the diagram

$$\begin{tikzcd} \pi_1(C-\delta) \arrow[dr, "\tilde{\rho}"'] & \pi_1(\mathbb{P}^2-\Delta) \arrow[l, "f_*"'] \arrow[d, "\rho\circ m"] \\ & PSL_2(\mathbb{C}) \end{tikzcd}$$

commutes.

Moreover, if some representation $\varrho : \Gamma \to SL_2(\mathbb{C})$ is such that $P \circ \rho : \Gamma \to SL_2(\mathbb{C})$, we will say that ϱ factors through an orbicurve.

Indeed, representations admitting such a factorisation can be obtained through pullback from the monodromy of some logarithmic flat connection on a curve [43]. Moreover, we have the following refinement by Loray, Pereira and Touzet [43] of a theorem by Corlette and Simpson [15].

Theorem 1.1.3 (Corlette–Simpson, Loray–Pereira–Touzet). *Let X be a quasi–projective surface. Then any non–rigid representation $\rho : \pi_1(X) \to PSL_2(\mathbb{C})$ with Zariski–dense image factors through a curve.*

Remark 1.1.4. *This implies that any representation satisfying condition **(C1)** that is neither rigid nor dihedral factors through a curve.*

1.1.2 The Zariski–Van Kampen method

What follows is a brief account of the classical Zariski–Van Kampen theorem in the case of complex projective plane curves. We refer the reader to a set of comprehensive lecture notes by Shimada [55] for further details.

Let $C \subset \mathbb{P}^2(\mathbb{C})$ be a projective plane curve ; consider a pair (x_0, L_0), where x_0 is a point in the complement of C in the projective plane $\mathbb{P}^2(\mathbb{C})$ and L_0 is a projective line which does not contain x_0. Then there is a natural projection

$$\pi_{x_0,L_0} : \mathbb{P}^2(\mathbb{C}) \setminus \{x_0\} \to L_0 \cong \mathbb{P}^1_{\mathbb{C}} \ .$$

There is a finite number of special fibres $L_1, \dots, L_n$ of the projection π_{x_0,L_0}, i.e fibres such that $\mathrm{Card}(L_i \cap C) < d := \deg(C)$. If one sets

$$X := \mathbb{P}^2(\mathbb{C}) \setminus \left(C \cup \bigcup_{i=1}^{n} L_i \right)$$

then the restriction $\tilde{\pi}_{x_0,L_0} := \pi_{x_0,L_0|_X}$ is a locally trivial fibration over $\mathbb{P}^1_n := \mathbb{P}^1_{\mathbb{C}} \backslash \{\pi_{x_0,L_0}(L_i) \,|\, i \in [n]\}$, with fibres equal to d–punctured Riemann spheres $\mathbb{P}^1_d$.

Theorem 1.1.5 (Van Kampen, Zariski). *Let X and M be quasi–projective manifolds such that M is aspherical (i.e $\pi_2(M) = 0$) and let $\pi : X \to M$ be a locally trivial fibration. Let $s : M \to X$ be a holomorphic section of π and fix $(p, x) \in M \times X$ such that $s(p) = x$. Then there is a group isomorphism*

$$\pi_1(X, x) \cong \pi_1(\pi^{-1}(p), x) \rtimes \pi_1(M, p) \ ,$$

where the semi–direct product is given by the monodromy of the fibration π.

This theorem is mainly a consequence of the homotopy exact sequence associated with the fibration π; indeed since M is aspherical the section s yields a splitting of this exact sequence, namely:

$$0 = \pi_2(M) \to \pi_1(\pi^{-1}(x), x) \to \pi_1(X, x) \to \pi_1(M, p) \to 0 \ .$$

Remark 1.1.6. *Note that in the case of the complement of a projective curve in $\mathbb{P}^2(\mathbb{C})$, there always exists such a section s, given by any generic line in the projective plane.*

Braid monodromy. In light of this result, one can see that the main technical difficulty in computing the fundamental group of X lies with the monodromy action of the fundamental group of the base over that of a generic fibre. Within the framework of Theorem 1.1.5, it can be described as follows: given a loop $\gamma : [0,1] \to M$ with base point p and another $g : [0,1] \to \pi^{-1}(p)$ with base point $x = s(p)$, the pointed fibres $(\pi^{-1}(\gamma(t)), s(\gamma(t)))$ naturally form a trivial fibre space over $[0,1]$. Therefore we can continuously deform the loop g into a loop

$$g_t : [0,1] \to \pi^{-1}(\gamma(t))$$

with base point $s(\gamma(t))$. We then define the aforementioned action via setting (see also [55]):

$$[g]^{[\gamma]} := [g_1] ,$$

where $[\cdot]$ denotes the homotopy class of a given loop.

In our case of study, the fundamental groups of both fibres and base are free groups that can be easily described. A special fibre only arises at some point p in the base if the line going through p and x_0 is either tangent to the curve C or passes through one of its singular points.

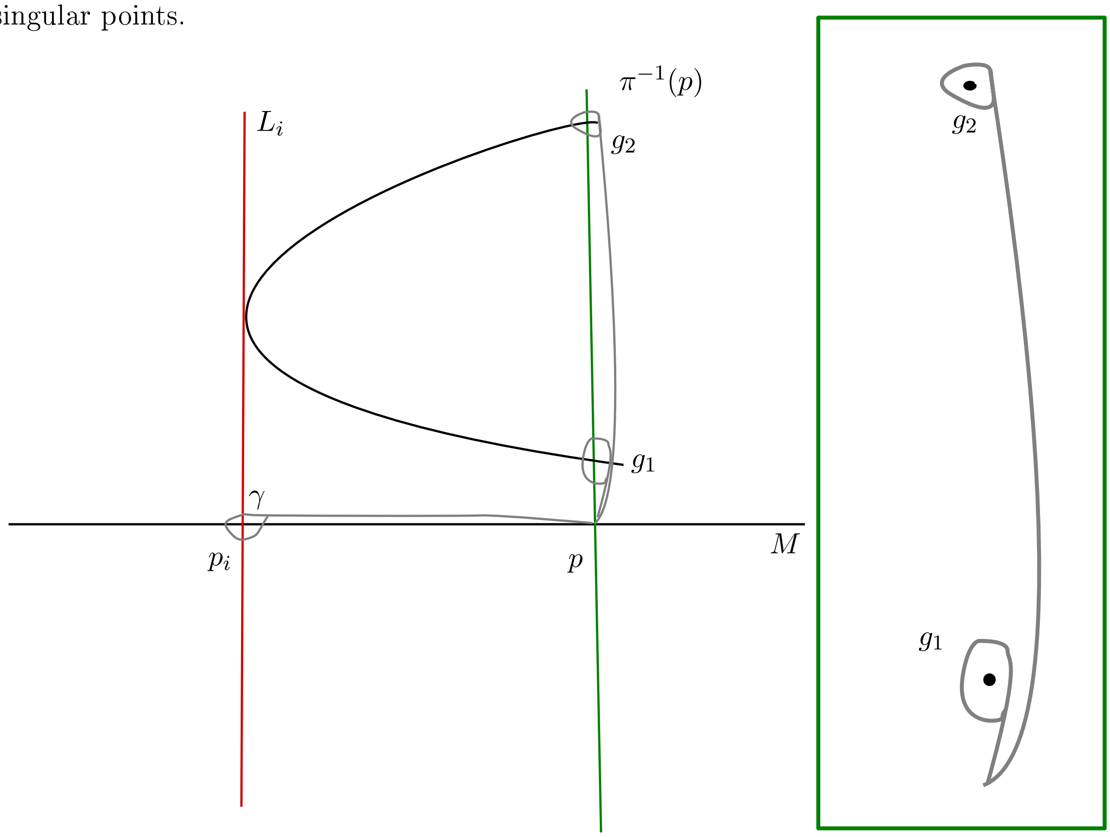

Figure 1.2: Monodromy around a tangent fibre.

The former case can be treated through explicit computations ; here we will only need to understand what happens when there a single tangency point of the form "$x = (y-1)^2$", as pictured in Fig. 1.2. Indeed,if $L_i = \pi^{-1}(p_i)$ is such a special fibre, then the only two loops on which the action of the loop γ going once around p_i in the base is going to act are those going around the local branches near the tangency, as pictured. If we call g_1 and g_2 those two loops, then a quick computation using local coordinates shows that:

$$[g_1]^{[\gamma]} = [\gamma^{-1} g_1 \gamma] = [g_2]$$

in the fundamental group of the total space (using the local section $y = 0$ to make the identification). So it remains to understand fibration monodromy (or braid monodromy) around curve singularities.

Lemma 1.1.7 ([55], p.14)**.** *Let Δ be some small bi–disk around the origin in $\mathbb{C}^2$ with coordinates (x, y) and consider the curve defined by the equation*

$$x^p - y^q = 0, \ \ \textit{for } p, q \geq 2\ .$$

Let $\mathrm{pr}_1 : \Delta \to \mathbb{C}$ be the first projection $(x, y) \mapsto x$ and fix some base point $b \in \mathrm{pr}_1(\Delta) \setminus \{0\}$; remark that $\pi_1(\mathrm{pr}_1(\Delta), b) \cong \mathbb{Z}$ and set γ to be some generator of this group. The fibre $\mathrm{pr}_1^{-1}(b)$ is equal to a disk in $\mathbb{C}$ punctured at the q–th roots of b^p and so its fundamental group is isomorphic to the free group $\mathbf{F}_q = \langle a_0, \ldots, a_{q-1}\rangle$ by identifying the a_i to some small circles around the aforementioned punctures to which we append a segment linking it to the base point (see Figure 1.3); in particular, this loop is homotopic to a circle going around the puncture once).Then the monodromy action of $\pi_1(\mathrm{pr}_1(\Delta), b)$ on $\pi_1(pr_1^{-1}(b), b)$ can be described as follows:

- *if $i + p < q$, then $a_i^\gamma = a_{i+p}$;*
- *else, if $i + p = eq + r$, with $r < q$, then*

$$a_i^\gamma = (a_{q-1} \ldots a_0)^e a_r (a_{q-1} \ldots a_0)^{-e}\ .$$

Example 1.1.8. *1. if $(p, q) = (2, 3)$ then the fundamental group of the fibre is equal to $\mathbf{F}_3 = \langle a_0, a_1, a_2\rangle$ (with the loops a_i chosen as above) and the loop γ acts as follows:*

- *$a_0^\gamma = a_2$;*
- *$a_1^\gamma = a_{1+2} = (a_2a_1a_0)a_0(a_2a_1a_0)^{-1}$;*
- *$a_2^\gamma = a_{2+2} = (a_2a_1a_0)a_1(a_2a_1a_0)^{-1}$.*

2. if $(p, q) = (5, 3)$ the the action can be described as follows:

- *$a_0^\gamma = a_{0+5} = (a_2a_1a_0)a_2(a_2a_1a_0)^{-1}$*
- *$a_1^\gamma = a_{1+5} = (a_2a_1a_0)^2 a_0(a_2a_1a_0)^{-2}$;*
- *$a_2^\gamma = a_{2+5} = (a_2a_1a_0)^2 a_1(a_2a_1a_0)^{-2}$;*

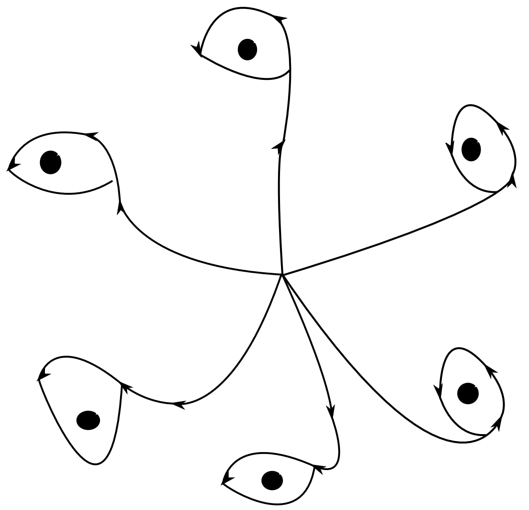

Figure 1.3: Choosing the generators a_i.

- $a_3^\gamma = a_{3+5} = (a_2a_1a_0)^2a_2(a_2a_1a_0)^{-2}$;
- $a_4^\gamma = a_{4+5} = (a_2a_1a_0)^3a_0(a_2a_1a_0)^{-3}$.

Patching singular fibres. This gives us an effective way of computing the fundamental group of the complement of C in $\mathbb{P}^2(\mathbb{C})$. More precisely, if one picks a generic fibre L of the projection $\pi_{x_0,L}$ one obtains the following presentation:

- one generator g_i for each intersection point of L and C;
- one generator γ_j for each singular fibre L_j of π_{x_0,L_0} apart from L_n;
- relations coming from the monodromy action (as described above) of the γ_j on the g_i;
- the relation $g_1 \dots g_d\gamma_1 \dots \gamma_{n-1} = 1$ if L_n is not a component of the curve C;
- relations given by the patching of singular fibres, as described below.

To patch a singular fibre L_j, we simply add a relation $\delta_j = 1$, where δ_j is a loop constructed as follows: set a family $(d_j)_j$ of small disjoint closed disks around the points $p_j = \pi_{x_0,L_0}(L_j)$ in the base L_0 and choose a line $L' \subset \mathbb{P}^2(\mathbb{C})$ such that $L' \cap \pi^{-1}_{x_0,L_0}(d_j)$ does not intersect the curve C. Then set δ_j to be the loop connecting $L' \cap \partial\pi^{-1}_{x_0,L_0}(d_j)$ to our base point through a fibre then γ_j.

Example 1.1.9. *To illustrate this method, we shall give a detailed description of the fundamental group of the complement of the curve C of type $3C_2$ illustrated in Figure 1.4. In homogeneous coordinates $[x : y : z]$, it is given by the equation:*

$$y(y-z)z(x^2-yz) = 0 .$$

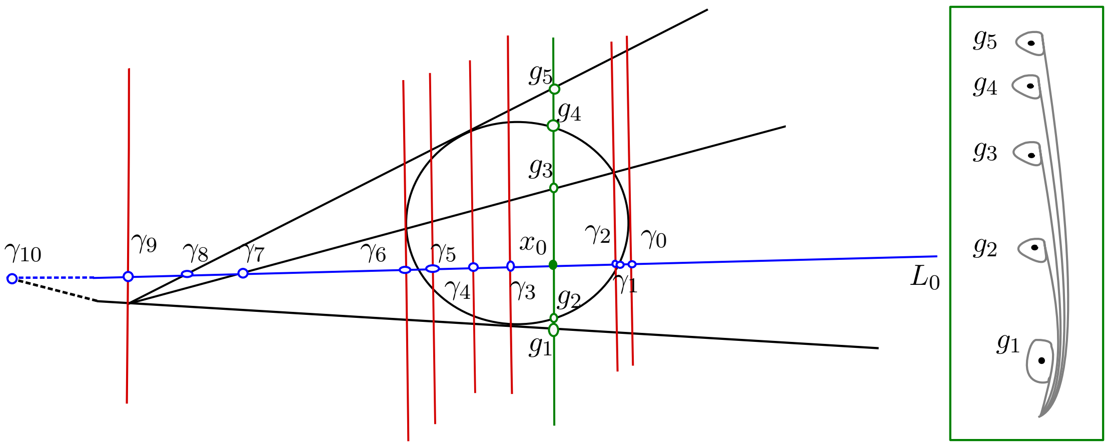

Figure 1.4: Example curve of type $C_2 \sqcup 3C_1$.

If we choose as a base point for our pencil of lines the point $x_0 := [0 : 1 : 0]$ *and set* $L_0 := \left(y = \frac{1}{2}\right)$*, we obtain the fibration "*$x =$*constant" of* $X = \mathbb{C}^2 \setminus \{y(y-1)(x^2-y) = 0\}$*. The fundamental group of the base (resp. any fibre) is a free group over 9 (resp. 4) generators* $\mathbf{F}_9 := \langle \gamma_1, \ldots, \gamma_{10} \,|\, \gamma_1 \ldots \gamma_9 = 1 \rangle$ *(resp.* $g_1, \ldots, g_5$*).*

Computing the braid monodromy around the loops γ_i*, one obtains the following relations (eliminating any redundancy):*

- $\gamma_2 g_2 \gamma_2^{-1} = g_4$;
- $\gamma_6 g_3 \gamma_6^{-1} = (g_1^{-1} g_2 g_1) g_3 (g_1^{-1} g_2 g_1)^{-1}$;
- $\gamma_0 g_2 \gamma_0^{-1} = g_3 g_2 g_3^{-1}$;
- $\gamma_9 g_1 \gamma_9^{-1} = (g_2 g_1 g_2)^{-1} g_1 (g_2 g_1 g_2)$.

By patching, these relations become:

- $g_2 = g_4$;
- $[(g_1^{-1} g_2 g_1), g_3] = 1$;
- $[g_2, g_3] = 1$;
- $(g_1 g_2)^2 = (g_2 g_1)^2$.

Therefore, we obtain that the fundamental group of the complement of C *is isomorphic to the group given by the following presentation:*

$$\langle a, b, c \,|\, [a, b] = [a, c^{-1} bc] = 1, (bc)^2 = (cb)^2 \rangle \; .$$

Using this method, Degtyarev [19] is able to give explicit presentations of all non–abelian groups arising as fundamental groups of complement of quintic curves in the complex projective plane.

1.2 Proof of Theorem A

1.2.1 Understanding the list

In [19], Degtyarev classifies the quintic curves in the projective plane $\mathbb{P}^2(\mathbb{C})$ whose complement has non–abelian fundamental group. In this work we are interested in infinite groups giving rise to representations satisfying conditions **(C1)** and **(C2)**, so we shall recall the parts of the aforementioned list of interest to us, starting by briefly recalling the notations used.

1.2.1.1 Groups appearing in the list

Toric groups. Toric group are the family of fundamental groups of toric links, defined as follows:

- for $r \geq 1$, set
$$T_{2,2r} := \langle a, b \,|\, (ab)^r = (ba)^r \rangle \ ;$$
- if p and q are two relatively prime integers set
$$T_{p,q} := \langle a, b \,|\, a^p = b^q \rangle \ .$$

Braid groups. Recall that the braid group on p strands is given by:

$$B_p := \langle \sigma_1, \ldots, \sigma_{p-1} \,|\, [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| > 1, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle \ .$$

"G–type groups". Let p be a prime number and $T \in \mathbb{Z}[t]$ be some integral polynomial; then we define the groups $G(T)$ and $G_p(T)$ as the extensions:

$$0 \to \mathbb{Z}[t]/(T) \to G(T) \to \mathbb{Z} \to 0$$

and

$$0 \to \mathbb{F}_p[t]/(T) \to G_p(T) \to \mathbb{Z} \to 0$$

where the conjugation action of the generator of the quotient on the kernel is the multiplication by t.

Note that these groups are solvable, therefore we will be able to use the following well–known result.

Proposition 1.2.1. *Let G be a solvable group. Then any irreducible group representation $\rho : G \to PSL_2(\mathbb{C})$ with infinite image is dihedral.*

Proof. Since G is solvable, then so is $\Gamma := \mathrm{Im}(\rho) \leq PSL_2(\mathbb{C})$. As such, it cannot be Zariski–dense either, therefore one gets that $\Gamma \leq \mathbb{D}_\infty$ by Proposition 1.1.1. □

Artin groups. We will also need a few Artin groups, namely:

- $\mathcal{A}^1(p,q,r) := \langle a,b,c \,|\, a^p = b^q = c^r = abc\rangle$;
- $\mathcal{A}^2(p,q,r) := \langle a,b,c \,|\, a^p = b^q = c^r = abc = 1\rangle$;
- $\mathcal{A}^3(p,q,r) := \langle a,b \,|\, a^p = b^q = 1, (ab)^r = (ba)^r\rangle$.

"Unusual" groups. Here we shall list some exceptional groups arising in Degtyarev's classification by giving finite presentations obtained through the Zariski–Van Kampen method.

- $\Gamma_5 := \langle u,v \,|u^3 = v^7 = (uv^2)^2\rangle$;
- $\Gamma_4 := \langle a,b,c \,|\, aba = bab,\, cbc = bcb,\, abcb^{-1}a = bcb^{-1}abcb^{-1}\rangle$;
- $\Gamma_3 := \langle a,b \,|\, [a^3,b] = 1\,, ab^2 = ba^2\rangle$;
- $\Gamma_3' := \langle a,b,c \,|\, aca = cac,\, [b,c] = 1,\, (ab)^2 = (ba)^2\rangle$;
- $\Gamma_2 := \langle a,b,c \,|\, [a,b] = [a,c^{-1}bc] = 1,\, (bc)^2 = (cb)^2\rangle$;
- $\Gamma_2' := \langle a,b,c \,|\, (ab)^2 = (ba)^2,\, (ac)^2 = (ca)^2,\, [b,c] = 1\rangle$.

1.2.1.2 Arnol'd's notation for curve singularities

In order to allow for an efficient listing of the curves involved in Degtyarev's classification we will make use of Arnol'd's notation for curves singularities [3]. More specifically:

- we will denote by A_p a singular point of local type
$$x^2 + y^{p+1} = 0 \quad ;$$
- we will denote by E_6 a singular point of local type
$$x^3 + y^4 = 0 \quad ;$$
- we will say that a curve is *of type* $C_{k_1} \sqcup \ldots \sqcup C_{k_n}$ if it (globally) has n irreducible components of respective degrees $k_1, \ldots, k_n$. In the case where several such components have the same degree, we will use the shorthand notation mC_{k_ℓ};
- if any degree d irreducible component of our curve has singular points, we shall denote it by writing $C_d(\Sigma)$, where Σ is the list of the aforementioned curve's singularities, of the form $m_1A_{p_1} \sqcup \ldots \sqcup m_nA_{p_n}$ (note that if the curve is smooth, we will simply use the shorthand C_d instead of $C_d(\emptyset)$);
- finally, we will use the following notation regarding the mutual position of two irreducible curves C and C':

- $\times d$ if C' intersects C with multiplicity d at a non–singular point of C;
- A_p if C' intersects C transversally at a singular point of C of type A_p;
- A_p^* if C' is tangent (with smallest possible multiplicity) to C transversally at a singular point of C of type A_p.

Moreover, if the curve is of type $C_3 \sqcup 2C_1$ and the two lines intersect at one of the special points described above, we shall underline it in the list.

Example 1.2.2. *A curve of type $C_3(A_1) \sqcup C_1$ with intersection $\times 2$, $\times 1$ would be made up of a nodal cubic and a line, the latter intersecting the former at two smooth points, once with a tangent and the other transversally. The same curve with intersection A_1, $\times 1$ would have the line intersect the cubic transversally at both some smooth point and the nodal singular point.*

1.2.1.3 The list

Table 1.1 sums up Degtyarev's list; for more details we refer the reader to the original paper [19]. Note that we have eliminated all finite groups from the list before reproducing it.

We shall go through Degtyarev's list in three steps:

- first, we investigate curves with large singularities, allowing us to remove some of them from the list;
- secondly, we eliminate any group in the list which cannot yield a representation satisfying conditions **(C1)** and **(C2)** for purely algebraic reasons;
- finally, we review the remaining fundamental groups in the list by curve type.

Doing so, we will make heavy use of the following elementary fact about $PSL_2(\mathbb{C})$.

Lemma 1.2.3. *Let $M \in PSL_2(\mathbb{C})$ centralising some non–abelian subgroup $G \leq PSL_2(\mathbb{C})$; then M is the identity element of $PSL_2(\mathbb{C})$.*

Proof. Let A, B be two elements in G such that $AB \neq BA$: A, B and M act on $\mathbb{P}^1(\mathbb{C})$ as Möbius transformations and since M commutes to A and B it stabilises (set–wise) both the sets F_A and F_B of fixed points for A and B. Moreover, up to changing A and/or B to A^2, B^2, M must have at least one common fixed point with both A and B, and since $\mathrm{Card}(F_A), \mathrm{Card}(F_B) \leq 2$ this means that M stabilises F_A and F_B point–wise. Now, as A and B do not commute, $\mathrm{Card}(F_A \cup F_B) \geq 3$, and so M must be diagonalisable; let us set $P \in PGL_2(\mathbb{C})$ such that PMP^{-1} is diagonal. Since G is non–abelian, there exists an element C in G such that PCP^{-1} is not diagonal; however $MC = CM$ thus M must be the identity element of $PSL_2(\mathbb{C})$. $\square$

Curve type	**Intersection type(s)**	**Group(s)**
$C_5(A_6 \sqcup 3A_2)$	—	Γ_5
$C_4(3A_2) \sqcup C_1$	$\times 2, \times 2$	Γ_4
	$\times 2, \times 1, \times 1$ or $A_2^*, \times 1$	B_3
	else	$G_3(t+1)$
$C_4(2A_2 \sqcup A_1) \sqcup C_1$	$\times 4$	B_4
	$\times 2, \times 2$	B_3
$C_4(2A_2) \sqcup C_1$	$\times 4$ or $\times 2, \times 2$	B_3
$C_4(A_4 \sqcup A_2) \sqcup C_1$	$\times 3, \times 1$	$\mathbb{Z} \times \mathcal{A}^2(2,3,5)$
	A_4^*	B_3
	$A_2, \times 2$	$G_5(t+1)$
$C_4(A_2 \sqcup A_2) \sqcup C_1$	$A_2, \times 2$	B_3
$C_4(A_6) \sqcup C_1$	$A_6, \times 2$	B_3
$C_4(A_5) \sqcup C_1$	$\times 4$ or $\times 2, \times 2$	B_3
$C_4(E_6) \sqcup C_1$	$\times 4$	$T_{3,4}$
	$\times 2, \times 2$	B_3
$C_3(A_2) \sqcup C_2$	$\times 3, \times 3$	Γ_3
$C_3(A_2) \sqcup 2C_1$	$\times 3\ ; \times 2, \times 1$	Γ_3'
	$\times 3\ ; A_2^*$	$T_{2,6}$
	$\underline{\times 3}\ ; \underline{\times 1}, A_2$	$T_{2,4}$
	$\underline{\times 3}\ ; \underline{\times 1}, \times 1, \times 1$	$\mathbb{Z} \times B_3$
	$\times 3\ ; A_2 \times 1$	$\mathbb{Z} \times B_3$
	$\times 3\ ; \times 1, \times 1, \times 1$	$\mathbb{Z} \times B_3$
	$\underline{\times 2}, \times 1\ ; \underline{\times 1}, \times 2$	$\mathbb{Z} \times B_3$
	$A_2, \underline{\times 1}\ ; \underline{\times 1}, \times 2$	$G(t^2-1)$
$C_3(A_1) \sqcup 2C_1$	$\times 3\ ; \times 3$	$G(t^3-1)$
	$\underline{\times 3}\ ; \underline{\times 1}, \times 2$	$G(t^2-1)$
	$\underline{\times 1}, \times 2\ ; \underline{\times 1}, \times 2$	$G(t^2-1)$
$2C_2 \sqcup C_1$	the two C_2 intersect with multiplicity 4	$\mathbf{F}_2$, $T_{2,4}$
	the two C_2 intersect at two points	$\mathbf{F}_2$, $T_{2,4}$
	the two C_2 intersect with multiplicity 3	$\mathbb{Z} \times B_3$
$C_2 \sqcup 3C_1$	the three C_1 have a common point	Γ_2, $\mathbb{Z} \times \mathbf{F}_2$
	else	Γ_2', $\mathbb{Z} \times \mathbf{F}_2$, $\mathbb{Z} \times T_{2,4}$
$5C_1$	the five C_1 have a common point	$\mathbf{F}_4$
	there is a quadruple point	$\mathbb{Z} \times \mathbf{F}_3$
	there are two triple points	$\mathbf{F}_2 \times \mathbf{F}_2$
	there is only one triple point	$\mathbb{Z} \times \mathbb{Z} \times \mathbf{F}_2$

Table 1.1: Degtyarev's list

1.2.2 Large singularities

The goal of this paragraph is to study the special cases where the quintic curve has some large order (*i.e* greater or equal to three) singular point. In fact, we prove that representations of the fundamental group of the complement of such curves almost always factor through some curve.

1.2.2.1 Five–fold singularities

First, we can eliminate the case of the curve $5C_1$ with a quintuple point through a rather elementary reasoning, as evidenced by the result below.

Proposition 1.2.4. *Let $\mathcal{Q}$ be a quintic in the projective plane $\mathbb{P}^2(\mathbb{C})$ of type $5C_1$ such that its five irreducible components share a common point. Then any representation $\rho : \pi_1(\mathbb{P}^2 - \mathcal{Q}) \to PSL_2(\mathbb{C})$ with non–abelian image factors through a curve.*

Proof. Since all irreducible components of $\mathcal{Q}$ intersect at a common point then by blowing it up we get a locally trivial fibration

$$\begin{array}{ccc} \hat{\mathbb{P}^2} - \hat{\mathcal{Q}} & \longleftarrow & \mathbb{C} \\ \Big\downarrow{\scriptstyle f} & & \\ B & & \end{array}$$

where B is isomorphic to $\mathbb{P}^1$ minus five points and $\hat{\mathcal{Q}}$ is the total transform of the quintic $\mathcal{Q}$. The homotopy exact sequence associated with f then yields:

$$0 = \pi_1(\mathbb{C}) \to \pi_1(\hat{\mathbb{P}^2} - \hat{\mathcal{Q}}) \to \mathbf{F}_4 = \pi_1(B) \to \pi_0(\mathbb{C}) \ .$$

Since $\pi_1(\mathbb{P}^2 - \mathcal{Q}) \cong \pi_1(\hat{\mathbb{P}^2} - \hat{\mathcal{Q}})$ (as we blew up a point in $\mathcal{Q}$) then one gets that $\pi_1(\mathbb{P}^2 - \mathcal{Q}) \cong \pi_1(B)$ thus ρ factors through B by means of the morphism f. $\square$

1.2.2.2 Quadruple singularities

In the rare case where a quintic curve in Degtyarev's list has a quadruple singularity, we can discard it using the following result.

Proposition 1.2.5. *Let $\mathcal{Q}$ be a quintic in the projective plane $\mathbb{P}^2(\mathbb{C})$ of type $5C_1$ such that exactly four of its irreducible components share a common point. Then any representation*

$$\rho : \pi_1(\mathbb{P}^2 - \mathcal{Q}) \to PSL_2(\mathbb{C})$$

with non–abelian image factors through a curve.

Proof. Up to an element in $PGL_3(\mathbb{C})$, one can assume that the line that does not intersect the other four at their common point is the line at infinity and that the aforementioned quadruple point is the origin of the corresponding affine chart (identified with $\mathbb{C}^2$). One gets a locally trivial fibration $\psi : \mathbb{P}^2 - \mathcal{Q} \to \mathbb{P}^1_4$ defined as follows: for any point $x \in \mathbb{P}^2 - Q$, $\psi(x)$ the line going through the origin and x. This fibration has fibre equal to $\mathbb{C}^*$ and a natural section given by any line not contained in $\mathcal{Q}$ nor going through the origin, the latter giving us an homeomorphism

$$\begin{aligned} f : \mathbb{P}^2 - Q &\xrightarrow{\sim} \mathbb{P}^1_4 \times \mathbb{C}^* \\ x &\mapsto (\psi(x), \tau(x)) \ , \end{aligned}$$

where $\tau(x)$ is the slope of the line $\psi(x) \subset \mathbb{C}^2$. This means that $\pi_1(\mathbb{P}^2 - Q) \cong \mathbf{F}_3 \times \mathbb{Z}$. The fact that the image of ρ must be non–abelian dictates that the image of one of these factors must also be. As such, any element in the image of the other one centralises a non–abelian subgroup in $PSL_2(\mathbb{C})$ and os must be trivial by Lemma 1.2.3. Thus one gets that ρ factors through $\mathbb{P}^1_4$ (as $\pi_1(\mathbb{C}^*) = \mathbb{Z}$ is abelian).

□

1.2.2.3 Triple singularities

It remains to see what happens when the studied curve has a singular point of order exactly three.

Proposition 1.2.6. *Let $\mathcal{Q}$ be a quintic in the projective plane $\mathbb{P}^2(\mathbb{C})$. Assume that there exists a triple point $p_0 \in \mathcal{Q}$,* i.e *that for any line L in $\mathbb{P}^2$ containing p_0 one has* $\mathrm{Card}(L \cap (\mathcal{Q} - p_0)) = 2$ *(counted with multiplicity); then any representation $\rho : \pi_1(\mathbb{P}^2 - \mathcal{Q}) \to PSL_2(\mathbb{C})$ with nonabelian, non–dihedral infinite image factors through a curve up to pull–back by a double covering.*

Proof. Start by blowing up p_0; in the following we will denote by $\hat{C}$ (resp. $\tilde{C}$) the total (resp. strict) transform of any curve $C \subset \mathbb{P}^2$ by this blow–up. This turns the pencil of lines going through the point p_0 into an actual fibration

$$\begin{tikzcd} \hat{\mathbb{P}^2} \arrow[d, "f"] & \mathbb{P}^1 \arrow[l] \\ B & \end{tikzcd}$$

endowed with a natural section τ given by the exceptional divisor $E(p_0)$ (note that both $E(p_0)$ and B are isomorphic to the projective line $\mathbb{P}^1$). Since a generic line containing p_0 in $\mathbb{P}^2$ cuts $\mathcal{Q} - p_0$ at two distinct points, we also get a "double section" of this fibration, namely a ramified double–covering $r : \mathcal{C} \xrightarrow{2:1} \mathbb{P}^1$ and a mapping $\sigma : \mathcal{C} \to \hat{\mathbb{P}^2}$ such that the diagram

$$\begin{tikzcd} & \hat{\mathbb{P}^2} \arrow[d, "f"] \\ \mathcal{C} \arrow[ur, "\sigma"] \arrow[r, "r"] & B \end{tikzcd}$$

commutes. Note that by construction $\sigma(\mathcal{C})$ is a component of the strict transform pf $\mathcal{Q}$; in the case where it is reducible we have a much stronger result (see Remark 1.2.7).

Consider the fibre product $S := \hat{\mathbb{P}^2} \times_B \mathcal{C}$; by definition, this is exactly the set

$$\{(p,q) \in \hat{\mathbb{P}^2} \times \mathcal{C} \,|\, f(p) = r(q)\},$$

giving us another commutative diagram (and a natural mapping $\chi : S \to \hat{\mathbb{P}^2}$):

$$\begin{tikzcd} S \arrow[r, "\chi"] \arrow[d, "\pi"] & \hat{\mathbb{P}^2} \arrow[d, "f"] \\ C \arrow[r, "r", "2:1"'] & B \end{tikzcd}$$

The second projection $\pi : S \to C$ yields a fibration

$$\begin{tikzcd} S \arrow[d, "\pi"] & \mathbb{P}^1 \arrow[l] \\ C & \end{tikzcd}$$

endowed with three distinct sections σ_0, σ_1 and σ_∞. Indeed one can define two sections of π using the double section σ as follows: for any $x \in C$ there exists $y \in C$ (generically distinct from x) such that $r(x) = r(y)$, so we set $\sigma_0(x) := (\sigma(x), x)$ and $\sigma_0(x) := (\sigma(y), x)$ (this is well defined since $f \circ \sigma = r$). The third section σ_∞ is then given by the exceptional divisor: $x \mapsto \tau \circ r(x)$. Note that there are finitely many points $x \in \mathcal{C}$ such that $\sigma_i(x) = \sigma_j(x)$ for some $i \neq j$ $(i, j = 0, 1, \infty)$; indeed these special points come from either singular points of $\mathcal{Q} - p_0$, fibres of f tangent to $\mathcal{Q}$ or possible intersections between $E(p_0)$ and the strict transform of $\mathcal{Q}$ in $\hat{\mathbb{P}^2}$.

One can now establish a birational mapping h between S and the direct product $\mathbb{P}^1 \times \mathcal{C}$. Let $x \in C$ be some generic point (so that $\sigma_0(x), \sigma_1(x)$ and $\sigma_\infty(x)$ are pairwise distinct) and define h on L as the homography sending $\sigma_0(x)$ (resp. $\sigma_1(x), \sigma_\infty(x)$) onto 0 (resp. $1, \infty$). This defines a rational mapping inducing an isomorphism between a Zariski–open set in S and one in $\mathcal{C} \times \mathbb{P}^1$.

Let us now look at the above recipe in topological terms: since we are blowing up a point in $\mathcal{Q}$ then $\pi_1(\mathbb{P}^2 - \mathcal{Q}) \cong \pi_1(\hat{\mathbb{P}^2} - \hat{\mathcal{Q}})$. Then remark that we can extend our representation ρ to the fundamental group of the complement of $\hat{\mathcal{Q}} \cup L_1 \cup \ldots \cup L_k$, where the L_i are the fibres of f that are either fibres above the ramification locus of r or contain points of the indeterminacy locus of h or h^{-1}, by setting the image of any simple loop around any special fibre L_i that is not a component of $\hat{\mathcal{Q}}$ to be the identity matrix I_2.

This means that we have a new representation $\tilde{\rho} : \pi_1(\hat{\mathbb{P}^2} - (\hat{\mathcal{Q}} \cup L_1 \cup \ldots \cup L_k) \to PSL_2(\mathbb{C})$ such that $\mathrm{Im}(\tilde{\rho}) = \mathrm{Im}(\rho)$. The covering r being non–ramified above $B - f(L_1 \cup \ldots \cup L_k)$ one gets that the fundamental group $G := \pi_1(S - \chi^*(\hat{\mathcal{Q}} \cup L_1 \cup \ldots \cup L_k))$ is isomorphic to an index at most two subgroup of $\pi_1(\hat{\mathbb{P}^2} - (\hat{\mathcal{Q}} \cup L_1 \cup \ldots \cup L_k))$. Thus, by restriction, one gets a representation $\rho_G : G \to PSL_2(\mathbb{C})$.

Since the birational map h only blows up and/or contract divisors contained in

$$\chi^*(\hat{\mathcal{Q}} \cup L_1 \cup \ldots \cup L_k),$$

there is an isomorphism between G and the fundamental group G' of $\mathcal{C} \times \mathbb{P}^1 - \delta$, where δ is the strict transform under h of $\chi^*(\hat{\mathcal{Q}} \cup L_1 \cup \ldots \cup L_k)$ ($i.e$ the Zariski–closure of its image under h). This divisor δ is by construction a finite union of "vertical" and/or "horizontal" curves in $\mathcal{C} \times \mathbb{P}^1$ thus $\mathcal{C} \times \mathbb{P}^1 - \delta = (\mathcal{C} - \eta) \times (\mathbb{P}^1 - \eta')$, where η (resp. η') is a finite divisorial sum of points in $\mathcal{C}$ (resp. $\mathbb{P}^1$). This in turn implies that $G' \cong \pi_1(\mathcal{C} - \eta) \times \pi_1(\mathbb{P}^1 - \eta')$.

Since $\mathrm{Im}(\rho_G)$ is an index at most 2 subgroup in the non–dihedral group $\mathrm{Im}(\rho)$, it must be non–abelian. This forces the image of one of the factors in G to be non–abelian therefore any element in the image of the other one centralises a non–abelian subgroup in $PSL_2(\mathbb{C})$ and so is trivial by Lemma 1.2.3. As such, ρ_G factors through either $\mathcal{C} - \eta$ or $\mathbb{P}^1 - \eta'$, using either the first or the second projection. □

Remark 1.2.7. *If the "double section" σ in the proof above is actually made up of two well–defined sections (i.e if $\sigma(\mathcal{C})$ is a reducible component of $\hat{\mathcal{Q}}$), then one naturally does not need the non–dihedral hypothesis. Indeed, $\sigma(\mathcal{C})$ must be made out of two lines and so we get three well–defined sections of the initial fibration, without needing to go through a double covering. Thus, any representation $\rho : \pi_1(\mathbb{P}^2 - \mathcal{Q}) \to PSL_2(\mathbb{C})$ with nonabelian, infinite image factors through a curve.*

The following curves have a triple singularity that falls under Remark 1.2.7, therefore we can remove them from the list:

- $C_4(E_6) \sqcup C_1$;
- $C_3(A_2) \sqcup 2C_1$ with intersection types $\times 3$; A_2^*, $\underline{\times 3}$; $\underline{\times 1}, A_2$ and $\underline{\times 1}, \times 2$; $\underline{\times 1}, A_2$;
- $2C_2 \sqcup C_1$ if either the two conics intersect with multiplicity 4 and the line is their common tangent at this point ($\mathbf{F}_2$–case in the list) or if they have two intersection points through which the line passes ($\mathbf{F}_2$–case);
- $C_2 \sqcup 3C_1$ if two of the line are tangent to the conic and the third passes through both tangency points;
- $5C_1$ with any number of triple points.

In the first two cases, one needs to chose the singular point of the irreducible component with highest degree as the singularity in Proposition 1.2.6. For example, when one looks at a curve of type $C_4(E_6) \sqcup C_1$ then one sees that any line going through the E_6 type singular point intersects the quintic in two distinguishable other points: one on the quartic C_4 and one on the line C_1. Thus, after blowing up the aforementioned singular point, the pencil of lines going through it becomes a fibration endowed with two sections. Thus Lemma 1.2.3

forces the image of one of the factors in G to be non–abelian therefore any element in the image of the other one centralises a non–abelian subgroup in $PSL_2(\mathbb{C})$ and so is trivial. As such, ρ_G factors through either $\mathcal{C} - \eta$ or $\mathbb{P}^1 - \eta'$, using either the first or the second projection.

To treat the case of $2C_2 \sqcup C_1$, take any intersection between the line and quadric and for $C_2 \sqcup 3C_1$ and $5C_1$ take any intersection between exactly three components. Note that this means that we can completely eliminate the free group on two generators $\mathbf{F}_2$ from the list.

1.2.3 Eliminating groups

In this paragraph, we eliminate several groups that, for strictly algebraic reasons, cannot give rise to a representation satisfying conditions (**C1**) and (**C2**). More precisely, we prove the following result.

Proposition 1.2.8. *Let G be one of the following groups:*

- *the braid groups B_3 and B_4;*
- *the group $G(t^3 - 1)$;*
- *the groups $G_p(t + 1)$ for some prime number p;*
- *the Artin group $\mathcal{A}^2(2, 3, 5)$;*
- *the "unusual" groups Γ_4, Γ_3 and Γ'_3;*
- *any direct product of $\mathbb{Z}$ and one of the above.*

*Let ρ be a representation of G into $PSL_2(\mathbb{C})$. Then ρ cannot satisfy both conditions **(C1)** and **(C2)**.*

1.2.3.1 Braid groups

On three strands. Consider the group $B_3 = \langle \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$; we prove the two following lemmas.

Lemma 1.2.9. *There is no representation $\rho : B_3 \to PSL_2(\mathbb{C})$ with non–abelian dihedral image.*

Proof. Let A (resp. B) be the image of σ_1 (resp. σ_2) by ρ. Then one has $ABA = BAB$ and thus $A = (AB)^{-1}B(AB)$ is conjugate to B. Moreover, one can easily check that $(AB)^3$ commutes with the whole (non abelian) image of ρ and so must be trivial (Lemma 1.2.3). Since ρ is dihedral then since its image must be non abelian one must have (up to global conjugacy and permuting A and B) either

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

or

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix}$$

for some $\lambda \in \mathbb{C} \setminus \{0, -1, 1\}$ (if $\lambda = \pm 1$ then the image of ρ would be abelian). In the first case, since A and B are conjugate, λ must be equal to $\pm i$ and so the groupe generated by A and B is finite abelian. In the second case, the condition one can only have $ABA = BAB$ if $\lambda = \pm 1$ *i.e* if $A = B$, meaning that the image of ρ must be abelian. $\square$

Lemma 1.2.10. *Let $\rho : B_3 \to PSL_2(\mathbb{C})$ be a representation with irreducible non–dihedral image. Then ρ cannot be rigid.*

Proof. As above, let A (resp. B) be the image of σ_1 (resp. σ_2) by ρ. Since ρ is not dihedral, then it is Zariski–dense and so must be rigid by Theorem 1.1.3. Up to global conjugacy, one can assume that either

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ or } A = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \text{ for some } u \in \mathbb{C} \setminus \{0, -1, 1\} \ .$$

In the first case, the facts that A and B are conjugate and $\mathrm{Im}(\rho)$ must be non–abelian forces

$$B = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$$

for some $u \in \mathbb{C}^*$, but then for any such u one has $ABA = BAB$ in $PSL_2(\mathbb{C})$ if and only if $u = -1$. However, one easily checks that for any $v \in \mathbb{C}^*$ the following defines a representation of B_3:

$$\sigma_1 \mapsto \begin{pmatrix} v & 1 \\ 0 & v^{-1} \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} v^{-1} & 0 \\ -1 & v \end{pmatrix}$$

and so ρ is not rigid. In the second case, one can check that the equation $B = (AB)^{-1}B(AB)$ (in the variable B) admits a solution for any $u \neq 0, \pm 1$ and that such a pair (A, B) defines a representation of B_3 for any such u. Therefore, the representation cannot be rigid. $\square$

The combination of Lemmas 1.2.9 and 1.2.10 with Theorem 1.1.3 yield that no representation $\rho : B_3 \to PSL_2(\mathbb{C})$ may satisfy conditions **(C1)** and **(C2)**.

On four strands. Consider the group

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \,|\, [\sigma_1, \sigma_3],\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \sigma_3\sigma_2\sigma_3 = \sigma_2\sigma_3\sigma_2 \rangle$$

and let A (resp. B, C) be the image of σ_1 (resp. σ_2, σ_3) by ρ; the braid relations then give us:

$$[A, C] = I_2, \quad ABA = BAB \quad \text{and} \quad BCB = CBC \ .$$

If either A or C is trivial then ρ factors through a representation of B_3 and so cannot satisfy conditions **(C1)** and **(C2)**. Moreover, B cannot be trivial since the image of ρ must be non–abelian, nor can A commute to B.

All three matrices must be conjugate as $B = (AB)^{-1}A(AB)$ and $C = (BC)^{-1}B(BC)$. As such, if ρ is dihedral then by applying Lemma 1.2.9 we get that both the groups $\langle A, B\rangle$ and $\langle B, C\rangle$ must be finite; combined with the fact that A and C commute, this forces $\mathrm{Im}(\rho)$ to be finite.

Else, we proceed again as above: up to global conjugacy, one can assume that either

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ or } A = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \text{ for some } u \in \mathbb{C} \setminus \{0, -1, 1\}\ .$$

In the first case, the facts that A, B and C are conjugate and $\mathrm{Im}(\rho)$ must be non–abelian forces (up to global conjugacy)

$$B = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$$

for some $u, v \in \mathbb{C}^*$, but then the relation $ABA = BAB$ (resp. $BCB = CBC$) in $PSL_2(\mathbb{C})$ if and only if $u = -1$ (resp. $v = 1$). However, one easily checks that for any $w \in \mathbb{C}^*$ the following defines a representation of B_4:

$$\sigma_1 \mapsto \begin{pmatrix} w & 1 \\ 0 & w^{-1} \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} w^{-1} & 0 \\ -1 & w \end{pmatrix}, \quad \sigma_3 \mapsto \begin{pmatrix} w & 1 \\ 0 & w^{-1} \end{pmatrix},$$

and so ρ is not rigid. In the second case, since A and C commute one must have

$$C = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \text{ for some } v \in \mathbb{C} \setminus \{0, -1, 1\}\ .$$

One can then check that both equations $B = (AB)^{-1}B(AB)$ and $B = (CB)C(CB)^{-1}$ (in the variable B) admits a solution for any $u, v \neq 0, \pm 1$ and that such a triple (A, B, C) defines a representation of B_4 for any such u, v. Therefore, the representation cannot be rigid.

1.2.3.2 Groups of type $G(T)$ and $G_p(T)$

Recall that since these groups are solvable, any of their irreducible representation into $PSL_2(\mathbb{C})$ must have dihedral image (see Proposition 1.2.1).

The group $G(t^3-1)$. Since $t^3-1=(t-1)(t^2+t+1)$, the group $G(t^3+1)$ is isomorphic to the semi–direct product $\mathbb{Z}^3 {}_{\varphi}\!\rtimes \mathbb{Z}$, where:

$$\varphi(n)\cdot\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}0&-1&0\\1&-1&0\\0&0&1\end{pmatrix}^n\begin{pmatrix}u\\v\\w\end{pmatrix}.$$

Fix the following set of generators for $G(t^3-1)$:

$$a:=\left(\begin{pmatrix}1\\0\\0\end{pmatrix},0\right),\ b:=\left(\begin{pmatrix}0\\1\\0\end{pmatrix},0\right),\ c:=\left(\begin{pmatrix}0\\0\\1\end{pmatrix},0\right),\ d:=\left(\begin{pmatrix}0\\0\\0\end{pmatrix},1\right)$$

and let A,B,C,D be their images under ρ. Then C centralises the whole image of ρ, which we assumed nonabelian therefore it must be equal to the identity in $PSL_2(\mathbb{C})$. Moreover, since A and B commute and $\mathrm{Im}(\rho)$ is dihedral and non–abelian one must have, up to global conjugacy:

$$A=\begin{pmatrix}u&0\\0&u^{-1}\end{pmatrix},\quad B=\begin{pmatrix}v&0\\0&v^{-1}\end{pmatrix}\quad\text{and}\quad D=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

or

$$A=B=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\quad\text{and}\quad D=\begin{pmatrix}u&0\\0&u^{-1}\end{pmatrix}$$

for some $u,v\in\mathbb{C}^*$. Note that A must be equal to B in the second case because those two matrices need to commute.

Using the above generators, one gets that in $G(t^3-1)$ one has the relation $d\cdot a=b\cdot d$ and so one must have $DA=BD$, which is only possible in the first case and if $uv=\pm1$. But one also has $d^2\cdot a=b\cdot d^2$ and so $A=B=I_2$ in $PSL_2(\mathbb{C})$, which is impossible as the image of ρ must be non–abelian.

Groups of type $G_p(t+1)$. Let p be some prime integer; then the group $G_p(t+1)$ is isomorphic to the semi–direct product $\mathbb{F}_p\,{}_{\varphi}\!\rtimes\mathbb{Z}$ where $\phi(n)\cdot\bar{k}=(-1)^n\bar{k}$. As such, it has two generators $a:=(\bar{1},0)$ and $b:=(\bar{0},1)$ such that (using multiplicative notation) $a^p=1$ and $ab^2=b^2a$.

If we consider a representation $\rho: G_p(t+1)\to PSL_2(\mathbb{C})$ and set $(A,B):=(\rho(a),\rho(b))$ then since B^2 commutes with every element of $\mathrm{Im}(\rho)$ we must have $A^p=B^2=I_2$, therefore the aforementioned image must be isomorphic to the semi–direct product $\mathbb{F}_p\rtimes\mathbb{F}_2$ and so is finite.

1.2.3.3 Group $\mathcal{A}^2(2,3,5)$

If one sets $A:=\rho(a)$, $B:=\rho(b)$ and $C:=\rho(c)$ then one gets $A^2=B^3=C^5=ABC=I_2$ in $PSL_2(\mathbb{C})$. Lifting this to $SL_2(\mathbb{C})$, one can thereofore assume that $A^2=\pm I_2$, $B^3=(AB)^5=-I_2$

(up to changing B to $-B$) and so $\mathrm{Tr}(A) \in \{-2, 0, 2\}$, $\mathrm{Tr}(B) = 2\cos(\pi/3) = 1$ and $\mathrm{Tr}(C) = 2\cos(\pi/5)$ or $2\cos(3\pi/5) = -2\cos(2\pi/5)$.

The study of hypergeometric equations by Schwartz's yielded a list of all triples of matrices (M_1, M_2, M_3) in $SL_2(\mathbb{C})$ satisfying $M_1M_2M_3 = I_2$ such that the group $\langle M_1, M_2, M_3\rangle \leq SL_2(\mathbb{C})$ is finite. An account of these works can be found in Chapter IX of [52], with the list itself being reproduced on p. 310. If $\mathrm{Tr}(A) = 2\cos(\pi/2) = 0$ then the aforementioned Schwartz's list gives us that the image of ρ is finite, isomorphic to $\mathfrak{A}_5$. Else, A must be equal to $\pm I_2$ and so $BC = \pm I_2$: $\mathrm{Im}(\rho)$ must be abelian.

"Unusual" groups

Several of the exceptional groups appearing in Degtyarev's list can be eliminated for algebraic reasons, as we show in this paragraph.

The group Γ_4 Consider the following group:

$$\Gamma_4 := \langle a, b, c \,|\, aba = bab,\ cbc = bcb,\ a(bcb^{-1})a = (bcb^{-1})a(bcb^{-1})\rangle\ .$$

Let ρ be a representation of the above group into $PSL_2(\mathbb{C})$ and set A (resp. B, C) to be the images $\rho(a)$ (resp. $\rho(b)$, $\rho(c)$). First remark that if $[B, C] = I_2$ in $PSL_2(\mathbb{C})$ then the second relation in the presentation above becomes:

$$C^2B = CB^2, \quad i.e \quad C = B$$

therefore ρ factors through a representation of the group:

$$\langle a, b \,|\, aba = bab\rangle$$

which is isomorphic to the braid group B_3 and so cannot satisfy both conditions **(C1)** and **(C2)**.

Lets assume then that B does not commute to C. This implies that the restriction $\tilde{\rho}$ of ρ to $\langle B, C\rangle$ factors through the braid group B_3, therefore it cannot have dihedral image by Lemma 1.2.9. So we get from Lemma 1.2.10 that there is an analytic family of matrices $u \mapsto B(u), C(u)$ containing B and C such that $C(u)B(u)C(u) = B(u)C(u)B(u)$ for any u in some Zariski–open set in $\mathbb{C}$. Therefore, $[A, B]$ must not be the identity or else one would have $A = B$ and so ρ would not be rigid. But then, one checks that, similarly to what we did in Lemma 1.2.10, the braid relations

$$AB(u)A = bab \quad \text{and} \quad A(B(u)C(u)B(u)^{-1})A = (B(u)C(u)B(u)^{-1})A(B(u)C(u)B(u)^{-1})$$

are compatible and so give us a way to analytically deform A as well, and so ρ is not rigid.

The group Γ_3 We are looking at representations of the group

$$\Gamma_3 := \langle a, b \,|\, [a^3, b] = 1 \,, ab^2 = ba^2 \rangle$$

into $PSL_2(\mathbb{C})$. Let ρ be such a representation and set A (resp. B) to be the images $\rho(a)$ (resp. $\rho(b)$), then Lemma 1.2.3 forces A^3 to be the identity and so up to conjugacy it must lift to the following matrix in $SL_2(\mathbb{C})$:

$$\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$$

where r is some root of the polynomial $Z^2 + Z + 1$. Then the relation $AB^2 = B^2 A$ and condition **(C1)** force B to lift to

$$\begin{pmatrix} \frac{r+2}{3} & 1 \\ -\frac{2}{3} & \frac{1-r}{3} \end{pmatrix} .$$

However, the triple $(A, B, (AB)^{-1})$ appears in Schwartz's list [52]; as a consequence, such a representation has finite image (in this case isomorphic to $\mathfrak{A}_4$).

The group Γ_3' This group is given by the following presentation:

$$\Gamma_3' := \langle a, b, c \,|\, aca = cac, \; [b, c] = 1, \; (ab)^2 = (ba)^2 \rangle \; .$$

Let ρ be a representation of the above group into $PSL_2(\mathbb{C})$ and set A (resp. B, C) to be the images $\rho(a)$ (resp. $\rho(b)$, $\rho(c)$). Condition **(C1)** and the fact that B and C commute force the pair (A, C) to be non–commutative. Since A and C are conjugate, it is possible to assume, up to conjugacy that:

$$A = \begin{pmatrix} u & 1 \\ 0 & u^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} v & 0 \\ w & v^{-1} \end{pmatrix} \text{ and } C = \begin{pmatrix} u & 0 \\ t & u^{-1} \end{pmatrix}$$

for some $u, v, w, t \in \mathbb{C}^*$. Solving the equations $ACA = CAC$ and $(AB)^2 = (BA)^2$ in $PSL_2(\mathbb{C})$ in the aforementioned variables, we get the following one–parameter families ($u \in \mathbb{C}^*$) of representations satisfying condition **(C1)**:

1. $$A = \begin{pmatrix} u^{-1} & 1 \\ 0 & u \end{pmatrix} \quad B = \begin{pmatrix} u^2 & 0 \\ -(u + u^{-1}) & u^{-2} \end{pmatrix} \quad C = \begin{pmatrix} u & 0 \\ -1 & u^{-1} \end{pmatrix} \; ;$$

2. $$A = \begin{pmatrix} u & 1 \\ 0 & u^{-1} \end{pmatrix} \quad B = \begin{pmatrix} u^2 & 0 \\ -\dfrac{(u^2+1)(u^4 - u^2 + 1)}{u^3} & u^{-2} \end{pmatrix} \quad C = \begin{pmatrix} u & 0 \\ -\dfrac{u^4 - u^2 + 1}{u^2} & u^{-1} \end{pmatrix} .$$

However, it is quite straightforward to check that these are not conjugate to representations with dihedral image: indeed if that were the case then in any pair of non–commutating matrices there should be one with eigenvalues $\pm i$. This can only happen here if u is equal to $\pm i$; this cannot happen in case 1. as the associated representation has reducible image, nor in case 2 since B would be equal to the identity. Consequently, it follows from Theorem 1.1.3 that all of the above factor through a curve.

1.2.3.4 Product groups of $\mathbb{Z}$ and one of the above

Let $\rho : \mathbb{Z} \times G \to PSL_2(\mathbb{C})$ be a group representation satisfying condition **(C1)** and set a to be a generator of the "$\mathbb{Z}$–part of the above product. Then $\rho(a)$ centralises the entire image of ρ, is a non–abelian subgroup of $PSL_2(\mathbb{C})$: therefore $\rho(a)$ is the identity element in $PSL_2(\mathbb{C})$ and so ρ factorises through a representation of G. It follows that there is no representation of $\mathbb{Z} \times G$ satisfying conditions **(C1)** and **(C2)**, as this would imply there exists one of G.

1.2.3.5 Filtered list

Curve type	**Intersection type(s)**	**Group(s)**
$C_5(A_6 \sqcup 3A_2)$	—	Γ_5
$C_3(A_1) \sqcup 2C_1$	$\underline{\times 3}$; $\underline{\times 1}, \times 2$	$G(t^2-1)$
	$\underline{\times 1}, \times 2$; $\underline{\times 1}, \times 2$	$G(t^2-1)$
$2C_2 \sqcup C_1$	the two C_2 intersect with multiplicity 4	$T_{2,4}$
	the two C_2 intersect at two points	$T_{2,4}$
$C_2 \sqcup 3C_1$	the three C_1 have a common point	Γ_2, $\mathbb{Z} \times \mathbf{F}_2$
	else	Γ_2', $\mathbb{Z} \times T_{2,4}$

Table 1.2: Degtyarev's list, after elimination of groups.

Proposition 1.2.8 allows us to substantially reduce Degtyarev's list, as evidenced in Table 1.2. It now only remains to look at this new list on a curve by curve basis to conclude the proof of Theorem A.

1.2.4 Remaining quintic curves and their fundamental group

1.2.4.1 Irreducible quintics

The only such curve with infinite non–abelian fundamental group is that of type $C_5(A_6 \sqcup 3A_2)$; the aforementioned group being isomorphic to

$$\Gamma_5 := \langle u, v \,|u^3 = v^7 = (uv^2)^2 \rangle \ .$$

This case has been previously studied by Cousin (see [17] Section 5.2, pages 110–112): any such representation factors through $10:1$ ramified cover over $\mathbb{P}^1$, and so cannot satisfy condition **(C2)**.

1.2.4.2 Curves of type $C_3(A_1) \sqcup 2C1$

We already eliminated the case where the intersection type is $\times 3, \times 3$ so we are left with $\underline{\times 3}; \underline{\times 1} \times 2$ and $\underline{\times 1} \times 2; \underline{\times 1} \times 2$. The fundamental group of the complement of both these curves is isomorphic to the solvable group $G(t^2 - 1)$ so ρ must be dihedral (see Proposition 1.2.1). This group is isomorphic to $\mathbb{Z}^2 {}_{\varphi}\!\rtimes \mathbb{Z}$, where:

$$\varphi(n) \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} u \\ v \end{pmatrix} ,$$

therefore it has three generators

$$a := \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} , 0 \right) \ , \ b := \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} , 0 \right) \quad \text{and} \quad c := \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} , 1 \right)$$

such that $b \cdot c^2 = c^2 \cdot b$, $ca = bc$ and $[a, b] = 1$. Thus if one sets A, B, C to be the images of these generators in $PSL_2(\mathbb{C})$, then one has necessarily $C^2 = I_2$. Since the image of ρ must be dihedral non–abelian, this implies that up to global conjugacy one gets diagonal A and B and:

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Therefore, the relation $CA = BC$ implies that there exists $u \in \mathbb{C}^*$ such that:

$$A = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} .$$

This is the third item in Theorem A.

1.2.4.3 Curves of type $2C_2 \sqcup C_1$

We already treated the "$\mathbf{F}_2$–cases" in Degtyarev's list using Remark 1.2.7, so we only need to concern ourselves with the remaining cases.

First, we consider the case where the two quadric curves have an intersection point of multiplicity 4 and the linear component is not the common tangent at the aforementioned point; an example of such a pair of curves is given (in homogeneous coordinates $[x : y : z]$ by the equation

$$(yz - x^2)(yz - x^2 - y^2) = 0 .$$

It is a straightforward application of the Zariski–Van Kampen method to show that the local monodromy around the linear component centralises the whole fundamental group. Indeed, if one looks at the loops given inFigure 1.5 then the braid monodromy relations

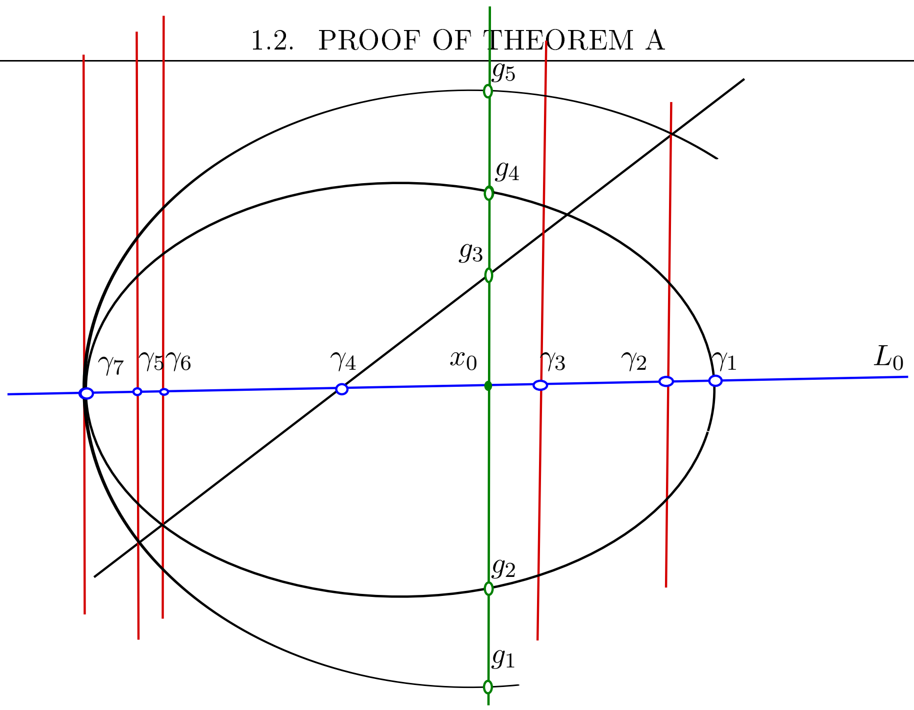

Figure 1.5: First case.

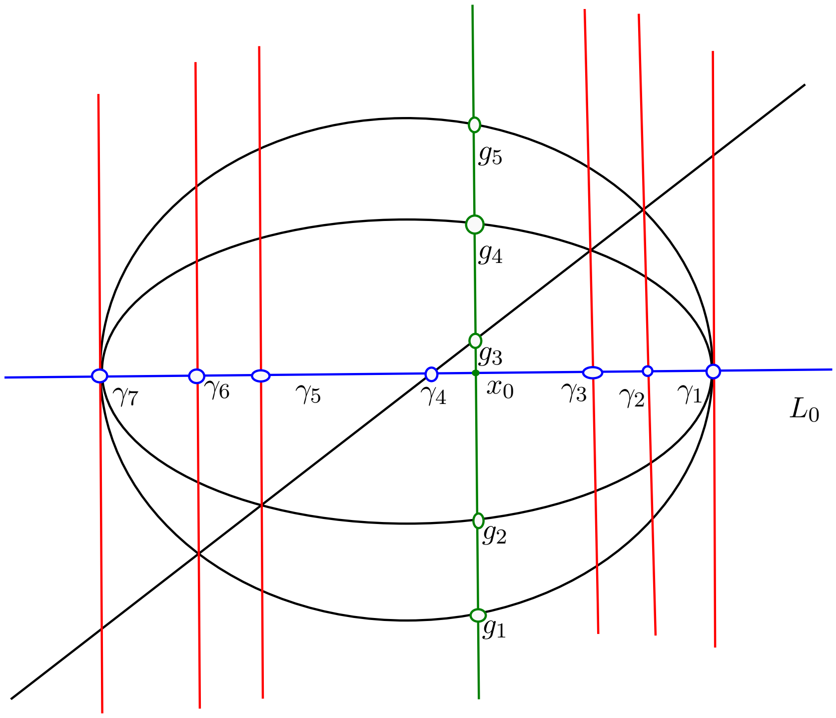

Figure 1.6: Second case.

yield that the fundamental group of the complement of this curve is generated by the loops g_1, g_2 and g_3 and that g_3 centralises the other two. Therefore, no such representation can satisfy conditions **(C1)** and **(C2)**.

The case where the two conics intersect at two points is treated in much the same way (see Figure 1.6); here again the local monodromy around the line, materialised by the loop g_3 must centralise the entire group and so be trivial, which contradicts conditions **(C1)** and **(C2)**.

1.2.4.4 Curves of type $C_2 \sqcup 3C_1$

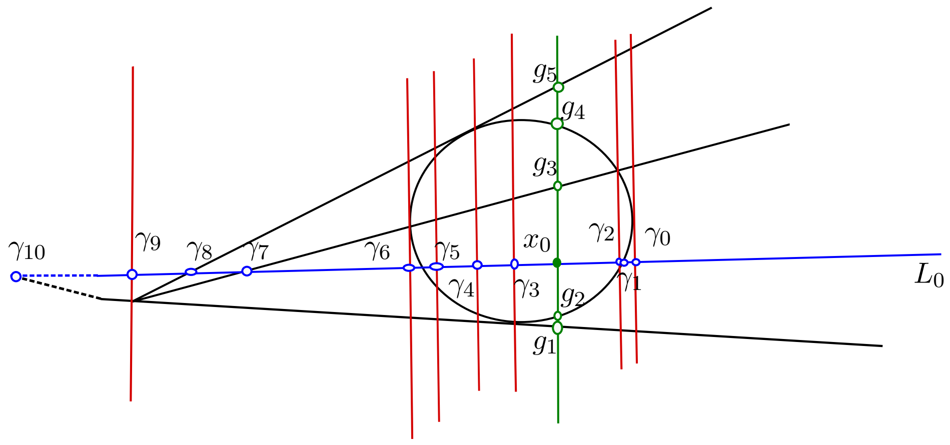

Figure 1.7: First case.

We already eliminated one of those when looking at triple singularities; let us now deal with the others.

Case 1: the three lines have a common point. If two of them are tangent to the conic then the fundamental group of the complement of this curve is given by:

$$\Gamma_2 := \langle a, b, c \,|\, [a, b] = [a, c^{-1}bc] = 1,\ (bc)^2 = (cb)^2 \rangle.$$

More precisely, if one applies the Zariski–Van Kampen method to this curve, one can identify a, b and c with (respectively) the loops g_3, g_2 and g_1 in Figure 1.7 (see Example 1.1.9).

If one sets $A := \rho(a)$, $B := \rho(b)$ and $C := \rho(c)$ then it follows from the fact that the local monodromy must be non–trivial and Lemma 1.2.3 that B must commute to $C^{-1}BC$. Thus, the commutator $[B, C^2]$ must be trivial, hence C^2 must be the identity element in $PSL_2(\mathbb{C})$ and since C cannot be trivial and $[A, B] = 1$ then up to global conjugacy ρ must be of the following type:

$$\rho : a \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ , \text{ for } u, v \in \mathbb{C}^*\ .$$

This is the second case in Theorem A.

On the other hand, if at most one of the lines is tangent to the conic, we derive from the Zariski–Van Kampen method that the local monodromy around one of the non–tangent

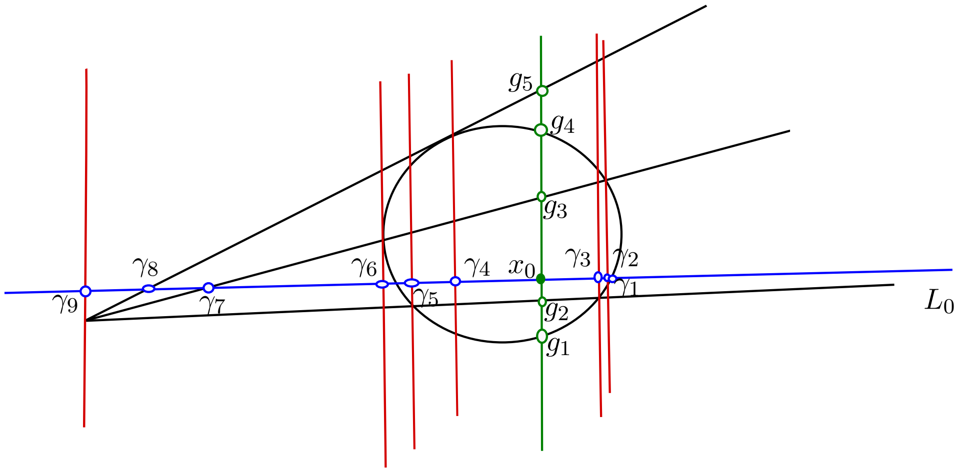

Figure 1.8: Second case.

lines must centralise the whole fundamental group. More precisely, braid monodromy relations from Figure 1.8 yield that the loop g_3 is in the centraliser of the fundamental group of the complement of the pictured curve.

Case 2: the three lines do not have a common point. In this case one must have at least two lines tangent to the conic in order to get a non–abelian fundamental group. If all three lines are tangent to the conic then the fundamental group of the complement is isomorphic to

$$\Gamma_2' := \langle a, b, c \,|\, (ab)^2(ba)^{-2} = (ac)^2(ca)^{-2} = [b, c] = 1\rangle \;.$$

More precisely, Degtyarev proves that we can take a (resp. b, c) to be a loop realising the local monodromy around the conic (resp. two of the tangent lines), as illustrated in Figure 1.9.

Since $\mathrm{Im}(\rho)$ must be non–abelian, either $C := \rho(c)$ or $B := \rho(b)$ does not commute to $A := \rho(a)$, say B. Then $(AB)^2 = (BA)^2$ and so $(AB)^2$ commutes to the non–abelian subgroup spanned by A and B in $PSL_2(\mathbb{C})$, therefore $(AB)^2$ must be equal to εI_2 for some $\varepsilon \in \{-1, 1\}$. This means that AB is diagonisable with eigenvalues in either $\{-1, 1\}$ (if $\varepsilon = 1$) or $\{-i, i\}$. In the former case, AB would be equal to $\pm I_2$ and so one would have $AB = BA$. Therefore, $(AB)^2$ must be equal to $-I_2$.

Up to conjugacy, one can assume that A, B and C are of the form

$$A = \begin{pmatrix} \alpha & \beta \\ -1 & \gamma \end{pmatrix} \quad , \quad B = \begin{pmatrix} \mu & \kappa \\ 0 & \mu^{-1} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \tau & \chi \\ 0 & \tau^{-1} \end{pmatrix} \;.$$

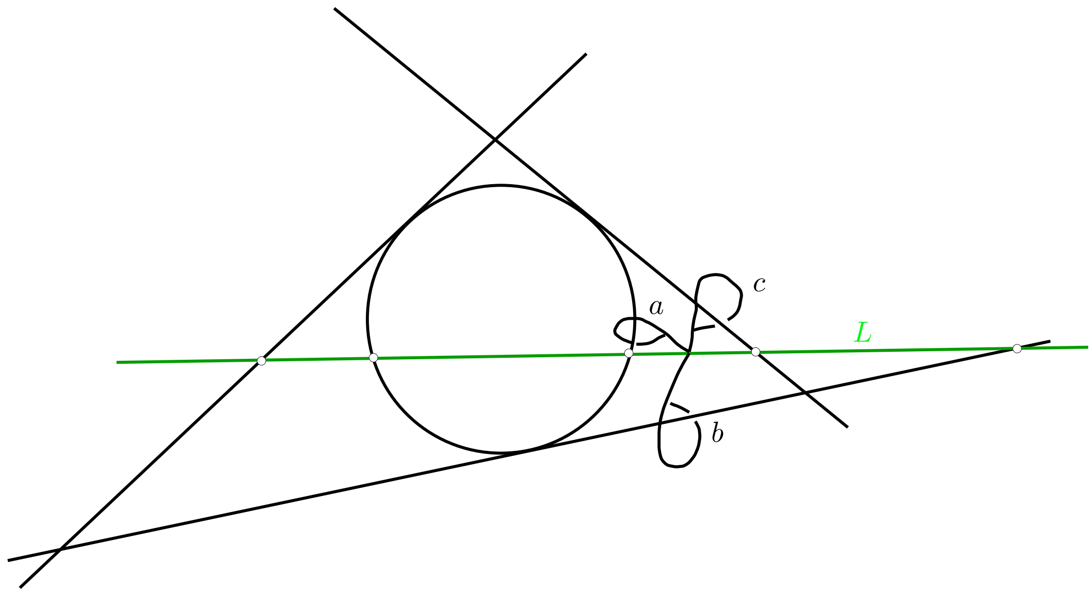

Figure 1.9: Third case

Since $(AB)^2 = -I_2$, $AB = -B^{-1}A^{-1}$ and so one must have

$$\begin{cases} \alpha\gamma + \beta &= \det(A) = 1 \\ \alpha\mu &= \gamma\mu^{-1} + \kappa \end{cases} . \tag{E1.1}$$

We assumed the monodromy representation to be non–degenerate; this means in particular that $ABAC = -B^{-1}A^{-1}AC = -B^{-1}C$ must not be equal to $\pm I_2$, *i.e* $B \neq \pm C$ and so $\mu^2 \neq \tau^2$.

Case 1: $\mu^2 \neq 1$. In this case, B is diagonalizable so it is possible (up to conjugacy) to assume $\kappa = 0$. Since B commutes to C, it follows that χ must also be zero and $\tau^2 \neq 1$. This implies that A does not commute to C and so one gets

$$\begin{cases} \alpha\gamma + \beta &= 1 \\ \alpha\mu^2 &= \gamma \\ \alpha\tau^2 &= \gamma \end{cases} . \tag{E1.2}$$

As $\tau^2 \neq \mu^2$, this forces α and γ to be zero, thus β must be one.

Case 2: $\mu^2 = 1$. Since B is not projectively trivial, then κ must be non–zero. The fact that B must commute to C forces τ^2 to be one and so one must also have $\chi \neq 0$. It is therefore impossible for A to commute to C and so by a similar reasoning to the one above, $(AC)^2 = -I_2$, thus one gets

$$\begin{cases} \alpha\gamma + \beta &= \det(A) = 1 \\ \alpha\mu &= \gamma\mu^{-1} + \kappa \\ \alpha\tau &= \gamma\tau^{-1} + \chi \end{cases} , \tag{E1.3}$$

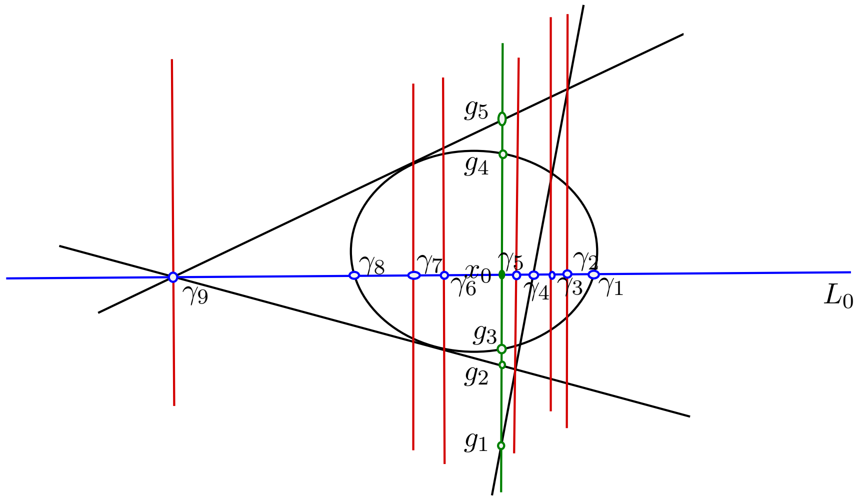

Figure 1.10: Fourth case.

which is equivalent to (since $\mu^2 = \tau^2 = 1$)

$$\left\{ \begin{array}{ccc} \alpha\gamma + \beta & = & \det(A) = 1 \\ \alpha & = & \gamma + \kappa\mu \\ \alpha & = & \gamma + \chi\tau \end{array} \right. , \tag{E1.4}$$

therefore $\kappa\mu = \chi\tau$. This means that $B = \pm C$ and so $ABAC = \pm I_2$, which contradicts conditions **(C1)** and **(C2)**.

In the end, the only (up to conjugacy) family of representations of Γ into $PSL_2(\mathbb{C})$ satisfying conditions **(C1)** and **(C2)** is as follows:

$$\rho_{u,v} : a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \text{ , for } u, v \in \mathbb{C}^* \text{ .}$$

This is the first case in Theorem A.

Otherwise, it is once again a consequence of braid monodromy relations that the local monodromy about any line not tangent to the conic must commute to every loop in the fundamental group (see Figure 1.10), thus we can discard this type of curve as well.

Chapter 2

Monodromy representations and mapping class group orbits

We saw in Chapter 1 that there are at most four families of "interesting" representations of fundamental group of complement of quintics into $PSL_2(\mathbb{C})$. The aim of this chapter is to explain how one can use this to obtain isomonodromic deformations of the five punctured sphere and to describe the associated mapping class group orbits.

2.1 Restricting a plane connection to generic lines

2.1.1 General method

Let ∇ be a rank two logarithmic flat $\mathfrak{sl}_2(\mathbb{C})$–connection over $\mathbb{P}^2(\mathbb{C})$ whose polar locus is exactly some quintic plane curve $Q \subset \mathbb{P}^2(\mathbb{C})$ and let $\rho : \pi_1(\mathbb{P}^2(\mathbb{C}) \setminus Q) \to SL_2(\mathbb{C})$ be its monodromy representation. Assume that the representation ρ is non–degenerate in the following sense.

Definition 2.1.1. *We say that the monodromy representation associated with a rank two logarithmic flat $\mathfrak{sl}_2(\mathbb{C})$–connection over $\mathbb{P}^2 - \mathcal{Q}$ is non–degenerate if*

- *its image forms an irreducible subgroup of $SL_2(\mathbb{C})$;*
- *its local monodromy around any irreducible component of $\mathcal{Q}$ is projectively non–trivial (i.e is non-trivial in $PSL_2(\mathbb{C})$).*

If the representation ρ factors through an orbicurve, then it is a known fact [15, 43] that ρ can be obtained as the monodromy of the pullback of some logarithmic flat connection over a curve. Isomonodromic deformations of punctured spheres arising from such constructions have been extensively studied by Diarra [24, 25], so let us assume that ρ does not factor through an orbicurve. Therefore, it follows from Theorem A that the pair $(\pi_1(\mathbb{P}^2(\mathbb{C}) \setminus Q), P \circ \rho)$ (where $P : SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$ is the canonical projection) must be one of the following:

1. $\pi_1(\mathbb{P}^2(\mathbb{C})\setminus Q) \cong \langle a,b,c\,|\,(ab)^2(ba)^{-2} = (ac)^2(ca)^{-2} = [b,c] = 1\rangle$,

$$\rho : a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \text{ , for some } u,v \in \mathbb{C}^* \text{ ;}$$

2. $\pi_1(\mathbb{P}^2(\mathbb{C})\setminus Q) \cong \langle a,b,c\,|\,[a,b] = [a,c^{-1}bc] = 1,\ (bc)^2 = (cb)^2\rangle$,

$$\rho : a \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ , for some } u,v \in \mathbb{C}^* \text{ ;}$$

3. $\pi_1(\mathbb{P}^2(\mathbb{C})\setminus Q) \cong \mathbb{Z}^2 \rtimes \mathbb{Z}$,

$$\rho : a \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ , for some } u \in \mathbb{C}^* \text{ .}$$

Let L be a generic line in the projective plane $\mathbb{P}^2(\mathbb{C})$; then L must intersect the quintic Q at exactly five points; identify L to $\mathbb{P}^1$ choosing a coordinate so that these are $0,1,\infty$ and some t_1,t_2. By restricting ∇ to L, we get a logarithmic flat connection ∇_L over the punctured Riemann sphere $\mathbb{P}^1_5 := \mathbb{P}^1(\mathbb{C})\setminus\{0,1,t_1,t_2,\infty\}$ whose monodromy ρ_L is defined through the following diagram:

$$\begin{tikzcd}
\pi_1(\mathbb{P}^1_5) \cong \mathbf{F}_4 \arrow[r, "\tau"] \arrow[dr, "\rho_L"'] & \pi_1(\mathbb{P}^2(\mathbb{C}) - Q) \arrow[d, "\rho"] \\
& SL_2(\mathbb{C})
\end{tikzcd}$$

where τ is the natural morphism given by restriction to L ; the Lefschetz hyperplane theorem (see [47], Theorem 7.4) shows that it is in fact onto. By construction, there exists a Zariski–open subset U of the dual projective space $\widehat{\mathbb{P}^2(\mathbb{C})}$ such that the family $(\nabla_L)_{L\in U}$ is an isomorphic deformation of the five punctured sphere.

Local monodromy Using the Zariski–Van Kampen method, it is actually quite straightforward to explicitly describe ρ_L in the three cases above. Indeed, if one denotes $\mathbf{F}_4 := \langle d_1,\ldots,d_4\,|\,\emptyset\rangle$ then the Lefschetz morphism τ is given by (respectively):

1.

$$\begin{aligned} d_1 &\mapsto b \\ d_2 &\mapsto a \\ d_3 &\mapsto bab^{-1} \\ d_4 &\mapsto c \text{ ;} \end{aligned}$$

2.

$$\begin{aligned} d_1 &\mapsto c \\ d_2 &\mapsto b \\ d_3 &\mapsto a \\ d_4 &\mapsto b\ ; \end{aligned}$$

3. (a)

$$\begin{aligned} d_1 &\mapsto b \\ d_2 &\mapsto ba \\ d_3 &\mapsto a \\ d_4 &\mapsto b^{-1}ab\ ; \end{aligned}$$

(b)

$$\begin{aligned} d_1 &\mapsto b \\ d_2 &\mapsto a \\ d_3 &\mapsto a \\ d_4 &\mapsto b^{-1}ab\ . \end{aligned}$$

Note that each of the images of the d_i correspond to the local monodromy around some irreducible component of the polar locus, in the following sense: let C be an irreducible curve contained in the polar locus of some logarithmic flat $\mathfrak{sl}_2(\mathbb{C})$–connection ∇ over $\mathbb{P}^2$, with associated monodromy representation ϱ. Set a point $p \in C$ such that no other irreducible curve in the polar locus of the connection passes through p; then if U is a sufficiently small analytic neighbourhood of p one gets:

$$\pi_1(U \setminus C \cap U) \cong \mathbb{Z}\ .$$

Let γ be any loop generating the above cyclic group; the conjugacy class of the matrix $\varrho(\gamma)$ does not depend on the choice of a base point for the fundamental group. Indeed, if γ is chosen as above for some base point q and if q' is some other point in the complement of the polar locus, then if one takes δ to be any path between q' and q, the loop $\delta \cdot \gamma \cdot \delta^{-1}$ is an element of the fundamental group of the complement based at q' whose monodromy is conjugate to $\varrho(\gamma)$.

Definition 2.1.2. *Using the notations above, define the local monodromy of ∇ around C as the conjugacy class of the matrix $\varrho(\gamma)$.*

As such, the restricted monodromy ρ_L must be given (up to global conjugacy) by the matrices appearing in Table 2.1.

Case	$x=0$	$x=1$	$x=t_1$	$x=t_2$	$x=\infty$
1.	$\begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$	$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & u^2 \\ -u^{-2} & 0 \end{pmatrix}$	$\begin{pmatrix} -uv^{-1} & 0 \\ 0 & -u^{-1}v \end{pmatrix}$
2.	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$	$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$	$\begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$	$\begin{pmatrix} 0 & -(uv^2)^{-1} \\ uv^2 & 0 \end{pmatrix}$
3. (a)	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & u^{-1} \\ -u & 0 \end{pmatrix}$	$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$	$\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}$	$\begin{pmatrix} -u & 0 \\ 0 & -u^{-1} \end{pmatrix}$
3. (b)	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$	$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$	$\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}$	$\begin{pmatrix} 0 & u^{-1} \\ -u & 0 \end{pmatrix}$

Table 2.1: Monodromy on a generic line.

Remark 2.1.3. *Note however that not all virtually abelian representations have finite mapping class group orbits. Indeed, this is even false in the four punctured case, as evidenced by Mazzocco's work on Picard's solutions of the Painlevé VI equation* [45].

2.1.2 Mapping class group orbits

The link between algebraic isomonodromic deformations of punctured sphere and finite orbits under the mapping class group action have been extensively studied in recent years [16, 27, 42]. In this paragraph, we describe the orbits associated with the isomonodromic deformations discussed earlier and show that only two of them are in fact distinct.

First let us fix some notations; the class of a representation $\rho : \mathbf{F}_4 = \langle d_1, \ldots, d_4 \,|\, \emptyset \rangle \to SL_2(\mathbb{C})$ in the $SL_2(\mathbb{C})$–character variety $\mathrm{Char}(0,5)$ of the five punctured sphere is fully determined by the following:

$$\begin{aligned} &t_1 := \mathrm{Tr}(\rho(d_1)),\ t_2 := \mathrm{Tr}(\rho(d_2)),\ t_3 := \mathrm{Tr}(\rho(d_3)), \\ &t_4 := \mathrm{Tr}(\rho(d_4)),\ t_5 := \mathrm{Tr}(\rho(d_1d_2d_3d_4)), \\ &r_1 := \mathrm{Tr}(\rho(d_1d_2)),\ r_2 := \mathrm{Tr}(\rho(d_1d_3)),\ r_3 := \mathrm{Tr}(\rho(d_1d_4)), \\ &r_4 := \mathrm{Tr}(\rho(d_2d_3)),\ r_5 := \mathrm{Tr}(\rho(d_2d_4)),\ r_6 := \mathrm{Tr}(\rho(d_3d_4)) \end{aligned}$$

and

$$r_7 := \mathrm{Tr}(\rho(d_1d_2d_3)),\ r_8 := \mathrm{Tr}(\rho(d_1d_2d_4)),\ r_9 := \mathrm{Tr}(\rho(d_1d_3d_4)),\ r_{10} := \mathrm{Tr}(\rho(d_2d_3d_4)).$$

We know from Cousin's work [16] that there is a correspondence between algebraic isomonodromic deformations of the five punctured sphere and finite orbits under the action of the pure mapping class group $\mathrm{PMod}(0,5)$ on the character variety $\mathrm{Char}(0,5)$. It is known that (see for example Section 9.3 in [29]) one has the following isomorphism (for any $n \geq 3$):

$$\mathrm{PMod}(0,n+1) \cong PB_n/Z(PB_n)\ ;$$

where PB_n is the pure braid group on n strand, i.e the kernel of the natural group homomorphism $B_n \twoheadrightarrow \mathfrak{S}_n$. As such, $\mathrm{PMod}(0,n+1)$ is an index $n!$ subgroup of the complete mapping class group $\mathrm{Mod}(0,n+1)$. The goal of this paragraph is to prove the following result.

Theorem B. *Consider the following four families of representations (parametrised by some $u,v,s \in \mathbb{C}^*$) of the free group over four generators $\mathbf{F}_4 := \langle d_1,\dots,d_4 \,|\, \emptyset\rangle$ into $SL_2(\mathbb{C})$.*

$$\rho_1 : d_1 \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1}\end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} 0 & u^2 \\ -u^{-2} & 0\end{pmatrix}$$
$$\rho_2 : d_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1}\end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1}\end{pmatrix}$$
$$\rho_3 : d_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} 0 & s^{-1} \\ -s & 0\end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} s & 0 \\ 0 & s^{-1}\end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} s^{-1} & 0 \\ 0 & s\end{pmatrix}$$
$$\rho_4 : d_1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} s & 0 \\ 0 & s^{-1}\end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} s & 0 \\ 0 & s^{-1}\end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} s^{-1} & 0 \\ 0 & s\end{pmatrix}$$

Then:

1. *the associated points in* $\mathrm{Char}(0,5)$ *give rise to four pairwise distinct families of length four finite orbits under the pure mapping class group* $\mathrm{PMod}(0,5)$*;*
2. *the families of (non–pure) mapping class group orbits associated with ρ_1 and ρ_2 are also distinct; however those associated with ρ_3 and ρ_4 are special cases of ρ_2–type orbits. More precisely, this means that for any $s \in \mathbb{C}^*$ and $i = 3,4$ there exist two parameters (u,v) (depending on s and i) such that the orbit of the class of ρ_i with parameter s is equal to that of ρ_2 with parameters (u,v).*

2.2 Proof of Theorem B

In order to prove Theorem B, we need to explicitly compute the orbits of the families of representations concerned. We have done so using a straightforward "brute–force" algorithm implemented in `Maple`. We make use of the following facts:

- the braid group on four strands

$$B_4 = \langle \sigma_1,\sigma_2,\sigma_2 \,|\, [\sigma_1,\sigma_3],\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \sigma_3\sigma_2\sigma_3 = \sigma_2\sigma_3\sigma_2\rangle$$

acts on the character variety $\mathrm{Char}(0,5)$ as follows: the braid σ_i sends the class of some representation $\rho : \mathbf{F}_4 \to SL_2(\mathbb{C})$ to that of the representation ρ^{σ_i} defined by:

$$\rho^{\sigma_i} : d_j \mapsto \begin{cases} \rho(d_i)\rho(d_{i+1})\rho(d_i)^{-1} \text{ if } j = i \\ \rho(d_i) \text{ if } j = i+1 \\ \rho(d_j) \text{ else} \end{cases} .$$

- The pure braid group PB_4 is generated by the following braids, for $1 \leq i < j \leq 3$:

$$\sigma_{i,j} := (\sigma_j \ldots \sigma_{i+1})\sigma_i(\sigma_j \ldots \sigma_{i+1})^{-1} ;$$

- the centre of the pure braid group on four strand is generated by the squared fundamental braid
$$\Delta^2 := (\sigma_3\sigma_2\sigma_1\sigma_2\sigma_1\sigma_1)^2 .$$
It is quite straightforward to see that Δ^2 acts on a representation ρ as the conjugacy by $\rho(d_1)\rho(d_2)\rho(d_3)\rho(d_4)$ and thus acts trivially on $\mathrm{Char}(0,5)$. Therefore the orbit of the class of ρ under the action of $\mathrm{PMod}(0,5)$ is the same as its orbit under that of PB_4.

We will now look in turn at the two conclusions of Theorem B by computing the associated orbits and using this data to reach the desired conclusion.

2.2.1 Orbits under the pure mapping class group

It follows from the remarks we made at the beginning of the proof that one only needs to compute the orbit of the classes of the representations $[\rho_i]$ under the pure braid group on four strand; which we achieve by using the "brute–force" algorithms written down in Appendix A.

In the coordinates $(\underline{t}\,|\,\underline{r})$ described earlier, the orbits of the classes of the representations $\rho_1, \ldots, \rho_4$ under the action of $\mathrm{PMod}(0,5)$ are as follows:

$[\rho_1]$

(a) $$\left(\frac{1+v^2}{v}, \frac{1+u^2}{u}, 0, 0, -\frac{u^2+v^2}{uv} \;\middle|\; \frac{1+u^2v^2}{uv}, 0, 0, 0, 0, -\frac{1+u^4}{u^2}, 0, 0, -\frac{u^4+v^2}{u^2v}, -\frac{1+u^2}{u}\right)$$

(b) $$\left(\frac{1+v^2}{v}, \frac{1+u^2}{u}, 0, 0, -\frac{u^2+v^2}{uv} \;\middle|\; \frac{u^2+v^2}{uv}, 0, 0, 0, 0, -\frac{u^4+v^4}{u^2v^2}, 0, 0, -\frac{u^4+v^2}{u^2v}, -\frac{u^2+v^4}{uv^2}\right)$$

(c) $$\left(\frac{1+v^2}{v}, \frac{1+u^2}{u}, 0, 0, -\frac{u^2+v^2}{uv} \;\middle|\; \frac{u^2+v^2}{uv}, 0, 0, 0, 0, -2, 0, 0, -\frac{1+v^2}{v}, -\frac{1+u^2}{u}\right)$$

(d) $$\left(\frac{1+v^2}{v}, \frac{1+u^2}{u}, 0, 0, \frac{u^2+v^2}{uv} \;\middle|\; \frac{1+u^2v^2}{uv}, 0, 0, 0, 0, -\frac{1+v^4}{v^2}, 0, 0, -\frac{1+v^2}{v}, -\frac{u^2+v^4}{uv^2}\right)$$

$[\rho_2]$

(a) $\left(0, \frac{1+v^2}{v}, \frac{1+u^2}{u}, \frac{1+v^2}{v}, 0 \,\middle|\, 0,0,0, \frac{1+u^2v^2}{uv}, \frac{1+v^4}{v^2}, \frac{1+u^2v^2}{uv}, 0,0,0, \frac{1+u^2v^4}{uv^2}\right)$

(b) $\left(0, \frac{1+v^2}{v}, \frac{1+u^2}{u}, \frac{1+v^2}{v}, 0 \,\middle|\, 0,0,0, \frac{1+u^2v^2}{uv}, 2, \frac{u^2+v^2}{uv}, 0,0,0, \frac{1+u^2}{u}\right)$

(c) $\left(0, \frac{1+v^2}{v}, \frac{1+u^2}{u}, \frac{1+v^2}{v}, 0 \,\middle|\, 0,0,0, \frac{u^2+v^2}{uv}, \frac{1+v^4}{v^2}, \frac{u^2+v^2}{uv}, 0,0,0, \frac{u^2+v^4}{uv^2}\right)$

(d) $\left(0, \frac{1+v^2}{v}, \frac{1+u^2}{u}, \frac{1+v^2}{v}, 0 \,\middle|\, 0,0,0, \frac{u^2+v^2}{uv}, 2, \frac{1+u^2v^2}{uv}, 0,0,0, \frac{1+u^2}{u}\right)$

$[\rho_3]$

(a) $\left(0,0, \frac{1+s^2}{s}, \frac{1+s^2}{s}, -\frac{1+s^2}{s} \,\middle|\, -\frac{1+s^2}{s}, 0,0,0,0,2, -\frac{1+s^4}{s^2}, -2, 0, 0\right)$

(b) $\left(0,0, \frac{1+s^2}{s}, \frac{1+s^2}{s}, -\frac{1+s^2}{s} \,\middle|\, -\frac{1+s^2}{s}, 0,0,0,0, \frac{1+s^4}{s^2}, -2, -2, 0, 0\right)$

(c) $\left(0,0, \frac{1+s^2}{s}, \frac{1+s^2}{s}, -\frac{1+s^2}{s} \,\middle|\, -\frac{(1+s^2)(1-s^2+s^4)}{s^3}, 0,0,0,0, \frac{1+s^4}{s^2}, -\frac{1+s^4}{s^2}, -\frac{1+s^4}{s^2}, 0, 0\right)$

(d) $\left(0,0, \frac{1+s^2}{s}, \frac{1+s^2}{s}, -\frac{1+s^2}{s} \,\middle|\, -\frac{1+s^2}{s}, 0,0,0,0,2, -2, -\frac{1+s^4}{s^2}, 0, 0\right)$

$[\rho_4]$

(a) $\left(0, \frac{1+s^2}{s}, \frac{1+s^2}{s}, \frac{1+s^2}{s}, 0 \,\middle|\, 0,0,0, \frac{1+s^4}{s^2}, 2, 2, 0,0,0, \frac{1+s^2}{s}\right)$

(b) $\left(0, \frac{1+s^2}{s}, \frac{1+s^2}{s}, \frac{1+s^2}{s}, 0 \,\middle|\, 0,0,0, \frac{1+s^4}{s^2}, \frac{1+s^4}{s^2}, \frac{1+s^4}{s^2}, 0,0,0, \frac{(1+s^2)(1-s^2+s^4)}{s^3}\right)$

(c) $\left(0, \frac{1+s^2}{s}, \frac{1+s^2}{s}, \frac{1+s^2}{s}, 0 \,\middle|\, 0,0,0,2,2, \frac{1+s^4}{s^2}, 0,0,0, \frac{1+s^2}{s}\right)$

(d) $\left(0, \frac{1+s^2}{s}, \frac{1+s^2}{s}, \frac{1+s^2}{s}, 0 \,\middle|\, 0,0,0,2, \frac{1+s^4}{s^2}, 2, 0,0,0, \frac{1+s^2}{s}\right)$.

One can then see just by checking the first four coordinates $t_1, \dots, t_4$ (corresponding to the traces of the matrices $\rho(d_1), \dots, \rho(d_4)$) that these constitute four distinct parametrised families of finite orbits, thus proving the first point of Theorem B.

2.2.2 Extended orbits

In order to show that these orbits are "truly different", we wish to know whether or not any two of them are contained in some orbit under the mapping class group action of $\mathrm{Mod}(0,5)$ over $\mathrm{Char}(0,5)$. First, we use the extended orbit computation algorithm showcased in Appendix A to actually compute the orbits of the classes of $\rho_1, \dots, \rho_4$ under the aforementioned action. Since these range in size from 40 to 240 elements we shall refer the reader to that particular appendix for the complete list, and we will restrict ourselves to giving the highlights.

The first thing to remark is that the orbit of ρ_1 (resp. ρ_2, ρ_3, ρ_4) is made up of 240 (resp. $120, 120, 40$) elements. This means that we have at least two distinct mapping class group orbits here : that of ρ_1, that of ρ_2 since they both have different cardinalities and the same number of free parameters. It now remains to see whether or not the family of orbits given by ρ_3 and ρ_4 are distinct from these two.

Looking at the list, one remarks that the fiftieth element in the orbit of ρ_2 is equal to

$$\left(0,0,\frac{1+v^2}{v},\frac{1+v^2}{v},\frac{1+u^2}{u},0,0,0,0,2,\frac{1+u^2v^2}{uv},\frac{u^2+v^2}{uv},0,0,\frac{1+u^2}{u}\right)\ .$$

A quick computation shows that the above becomes, after the change of parameters $u \mapsto -s$, $v \mapsto s$

$$\left(0,0,\frac{1+s^2}{s},\frac{1+s^2}{s},-\frac{1+s^2}{s},0,0,0,0,2,-\frac{1+s^4}{s^2},-2,0,0,-\frac{1+s^2}{s}\right)$$

which is actually the first point in the extended orbit of ρ_3. Therefore, these two orbits must be equal. Moreover, the one–hundred and fourth element in the orbit of ρ_2 is equal to

$$\left(0,\frac{1+u^2}{u},\frac{1+v^2}{v},\frac{1+v^2}{v},0,0,0,\frac{1+u^2v^2}{uv},\frac{u^2+v^2}{uv},2,0,0,0,\frac{1+u^2}{u},0\right)\ .$$

Here again, an adequate change of parameters (namely $u, v \mapsto s$ turns it into the first element in the orbit of ρ_4

$$\left(0,\frac{1+u^2}{u},\frac{1+u^2}{u},\frac{1+u^2}{u},0,0,0,\frac{1+u^4}{u^2},2,2,0,0,0,\frac{1+u^2}{u},0\right)$$

meaning that this one is just a special case of that of ρ_2.

Remark 2.2.1. *1. A natural follow–up question to the results presented here could be the following: can this procedure yield any more algebraic Garnier solutions ? More precisely, we have seen in Section I1.2 that the Garnier system associated to an isomonodromic deformation is determined by the traces of its local monodromy, or to put it another way, the coordinates $t_1, \ldots, t_5$ in the character variety. Any one of our two–parameter families of representations ρ_1 and ρ_2 gives us a mapping*

$$\begin{aligned} f : \mathbb{C}^* \times \mathbb{C}^* &\to \mathbb{C}^5 \\ (u, v) &\mapsto (t_1, \ldots, t_5)\ . \end{aligned}$$

The fibre of f above any $\underline{t}$ parametrises a family of algebraic solutions of the associated Garnier system. Indeed, if $(u, v) \in f^{-1}(\underline{t})$ then the orbit of $(\underline{t}\,|\,\underline{r}(u, v)) \in \mathrm{Char}(0, 5)$ orbit under the pure mapping class group $\mathrm{PMod}(0, 5)$ corresponds to such a solution. Therefore, finding (u', v') in the fibre such that $\underline{r}(u', v') \neq \underline{r}(u, v)$ would yield another one. However, here in both cases we have fibres of the form $\{(u, v), (u^{-1}, v^{-1})\}$ and the orbits of $[\rho_1]$ and $[\rho_2]$ are invariant under $(u, v) \mapsto (u^{-1}, v^{-1})$.

2. In the same line of thought, one could look at families of Garnier systems equivalent under symmetries of the Schlesinger equation. More precisely, consider the *symmetrised space*

$$\tau : \mathbb{C}^5 \twoheadrightarrow \mathbb{C}^{(5)} := \mathbb{C}^5/\mathfrak{S}_5$$

and try to find new solutions parametrised by the fibres of

$$\varphi := \tau \circ f : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^{(5)}.$$

- *For* $[\rho_1]$, φ *is defined by*

  $$(u,v) \mapsto \left[\left(\frac{1+u^2}{u}, \frac{1+v^2}{v}, -\frac{u^2+v^2}{uv}, 0, 0\right)\right] = \left[\left(u+\frac{1}{u}, v+\frac{1}{v}, -\left(\frac{u}{v}+\frac{v}{u}\right), 0, 0\right)\right]$$

  and so its fibres are of the form

  $$\left\{(u,v)\,,\, \left(\frac{1}{u},\frac{1}{v}\right),\, (v,u)\,,\, \left(\frac{1}{v},\frac{1}{u}\right),\, \left(-\frac{u}{v},\frac{1}{v}\right),\, \left(-\frac{v}{u},v\right),\, \left(\frac{1}{u},-\frac{v}{u}\right),\, \left(u,-\frac{u}{v}\right)\right\}\ .$$

  Using our explicit computation of the orbits under the complete mapping class group $\mathrm{Mod}(0,5)$ *we check that they are invariant under any of the above change of variables, and so this procedure can not yield any extra algebraic Garnier solution.*

- *The function* φ *associated with* $[\rho_2]$ *is given by*

  $$(u,v) \mapsto \left[\left(\frac{1+u^2}{u}, \frac{1+v^2}{v}, \frac{1+v^2}{v}, 0, 0\right)\right] = \left[\left(u+\frac{1}{u}, v+\frac{1}{v}, v+\frac{1}{v}, 0, 0\right)\right]$$

  and so its fibres are of the form

  $$\left\{(u,v)\,,\, \left(\frac{1}{u},\frac{1}{v}\right),\, \left(\frac{1}{u},v\right),\, \left(u,\frac{1}{v}\right)\right\}\ .$$

  Here again we check that we cannot get anything more in this way.

Chapter 3

First family of solutions

In this chapter, we describe the main result of the paper [32], namely an explicit construction of a two–parameter family of logarithmic flat connections over the complement of a particular quintic curve in $\mathbb{P}^2_{\mathbb{C}}$. The restriction of any element of this family to generic lines in the projective plane gives a isomonodromic deformation over the five punctured sphere, to which we can associate an algebraic solution of some Hamiltonian system of partial differential equations, namely the Garnier-2 system.

3.1 Setup and main results

3.1.1 Topology of the complement of a particular plane quintic

We concern ourselves with setting up a two–parameter family of logarithmic flat $\mathfrak{sl}_2(\mathbb{C})$–connections over $\mathbb{P}^2$ with a specific polar locus, namely a quintic curve $\mathcal{Q}$ composed of a circle and three tangent lines. More precisely, in homogeneous coordinates $[x : y : t]$, $\mathcal{Q}$ is defined, up to $PGL_3(\mathbb{C})$ action, by the equation

$$xyt(x^2 + y^2 + t^2 - 2(xy + xt + yt)) = 0 \ .$$

Before stating our main result, let us specify what we are looking for: we want to find a family of rank two logarithmic flat connections over $\mathbb{P}^2$ with polar locus equal to some small degree curve and "interesting monodromy". We will show that it is possible to do so with the quintic $\mathcal{Q}$ defined above.

Definition 3.1.1. *We say that the monodromy representation associated with a rank two logarithmic flat $\mathfrak{sl}_2(\mathbb{C})$–connection over $\mathbb{P}^2 - \mathcal{Q}$ is non–degenerate if*

- *its image forms an irreducible subgroup of $SL_2(\mathbb{C})$;*
- *its local monodromy around any irreducible component of $\mathcal{Q}$ is projectively non–trivial (i.e is non-trivial in $PSL_2(\mathbb{C})$).*

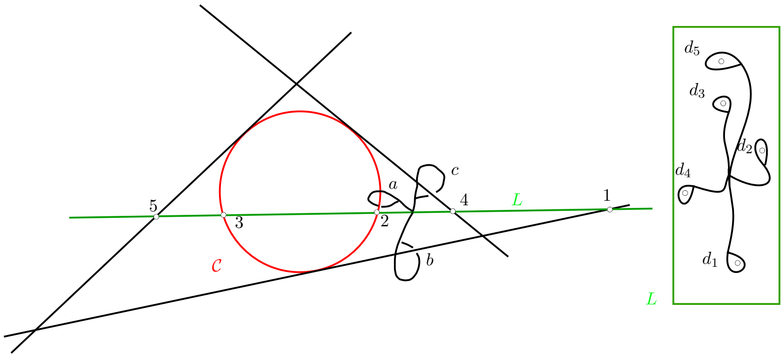

Figure 3.1: Fundamental group of the complement of the quintic $\mathcal{Q}$ in $\mathbb{P}^2$ and restriction to a generic line.

Note that if the projective monodromy of such a connection satisfies the conditions **(C1)** and **(C2)** introduced in Chapter 1 then it lifts to a non–degenerate representation. However, the converse does not hold, as we will show in Theorem D.

We know from Degtyarev's list that the fundamental group of the complement of a smooth conic and three tangent lines in $\mathbb{P}^2$ is isomorphic to the following group

$$\Gamma_2 := \langle a, b, c \,|\, (ab)^2(ba)^{-2} = (ac)^2(ca)^{-2} = [b, c] = 1 \rangle \ .$$

More precisely, Degtyarev proves that we can take a (resp. b, c) to be a loop realising the local monodromy around the conic $\mathcal{C} := (x^2 + y^2 + t^2 - 2(xy + xt + yt) = 0)$ (resp. the lines $(y = 0)$, $(x = 0)$), as illustrated in the left–hand side of Figure 3.1. Also note that the fundamental group of the intersection of $\mathbb{P}^2 - \mathcal{Q}$ with any generic line is isomorphic to the free group $\mathbf{F}_4 := \langle d_1, \ldots, d_5 \,|\, d_1 \ldots d_5 = 1\rangle$; the Lefschetz hyperplane theorem (see [47], Theorem 7.4) tells us that the natural morphism $\tau : \mathbf{F}_4 \to \Gamma_2$ is onto. Moreover we know from the explicit Zariski–Van Kampen method given in Subsection 4.1 of [19] that the group Γ_2 can be computed by taking the four free generators of the fundamental group of the intersection of $\mathbb{P}^2 - \mathcal{Q}$ with any generic line and adding some braid monodromy relations. Thus, if we chose a line going through the base point used to define a, b and c then τ is given (up to a permutation of the d_i) by (see the right–hand side of Figure 3.1):

$$\begin{aligned} d_1 &\mapsto b \\ d_2 &\mapsto a \\ d_3 &\mapsto bab^{-1} \\ d_4 &\mapsto c \\ d_5 &\mapsto (abac)^{-1} \ . \end{aligned}$$

In particular, any non–degenerate representation ρ of Γ_2 must satisfy

$$\rho(a), \rho(b), \rho(c), \rho(abac) \neq \pm I_2 .$$

Proposition 3.1.2. *The only (up to conjugacy) family of non–degenerate representations of Γ_2 into $SL_2(\mathbb{C})$ is as follows:*

$$\rho_{u,v} : a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} , \text{ for } u, v \in \mathbb{C}^* .$$

This result has already been proven in Chapter 1: indeed when we looked at representation of Γ satisfying conditions **(C1)** and **(C2)** we only used the fact that any such group homomorphism must be non–degenerate in the sense of Definition 3.1.1.

Remark 3.1.3. *This implies that any non–degenerate representation of Γ will have a "sizeable" kernel; indeed recall that we have a natural two–fold ramified cover $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{2:1} \mathbb{P}^2$ ramifying over the diagonal Δ. This mapping yields a nonramified cover $\tilde{\pi} : X - D \xrightarrow{2:1} \mathbb{P}^2 - \mathcal{Q}$, where $X := \mathbb{P}^1 \times \mathbb{P}^1$, and thus one gets that $\pi_1(X - D)$ embeds into $\pi_1(\mathbb{P}^2 - \mathcal{Q}) \cong \Gamma$ as an index two subgroup. If one denotes by $\mathbb{P}^1_n$ the n punctured sphere, the projection on the line $(y = 0)$ gives a fibration $X - D \to \mathbb{P}^1_3$ with fibre $\mathbb{P}^1_4$. As the universal cover of $\mathbb{P}^1_3$ (namely the hyperbolic plane $\mathbb{H}^2$) is contractible, the homotopy exact sequence associated with this fibration yields:*

$$0 = \pi_2(\mathbb{P}^1_3) \to \pi_1(\mathbb{P}^1_4) \to \pi_1(X - D) \to \pi_1(\mathbb{P}^1_3) \to 0 .$$

In particular, there is an injective morphism from $\pi_1(\mathbb{P}^1_4) \cong \mathbf{F}_3$, where $\mathbf{F}_r$ denotes the free group over r generators, into $\pi_1(X - D)$; which in turn implies that the group Γ contains a noncommutative free group.

Moreover the orbifold fundamental group $\Gamma^{\mathrm{orb}}_\pi$ associated with the ramified cover π also contains a free group; more precisely if we define

$$\Gamma^{\mathrm{orb}}_\pi := \langle a, b, c \,|\, (ab)^2(ba)^{-2} = (ac)^2(ca)^{-2} = [b, c] = a^2 = 1 \rangle$$

then π induces an embedding of the fundamental group of X minus six lines into $\Gamma^{\mathrm{orb}}_\pi$, i.e $\mathbf{F}_2 \times \mathbf{F}_2 \hookrightarrow \Gamma^{\mathrm{orb}}_\pi$. This is especially relevant since the projective representations $\Gamma \to PSL_2(\mathbb{C})$ associated with the monodromy of the connections we will describe in this paper factor through this orbifold fundamental group.

3.1.2 Main results

The core of this chapter will be devoted to proving the following theorem, in which we explicitly construct the announced family of rank two logarithmic flat connections.

Theorem C. *There exists an* explicit *two–parameter family $\nabla_{\lambda_0,\lambda_1}$ of logarithmic flat connections over the trivial rank two vector bundle $\mathbb{C}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ with the following properties:*

(i) the polar locus of $\nabla_{\lambda_0,\lambda_1}$ is equal to the quintic $\mathcal{Q} \in \mathbb{P}^2$ defined by the equation

$$xyt(x^2 + y^2 + t^2 - 2(xy + xt + yt)) = 0 \ ;$$

(ii) the monodromy of $\nabla_{\lambda_0,\lambda_1}$ is conjugated to $\rho_{u,v}$ with $u = -e^{-i\pi\lambda_0}$ and $v = e^{-i\pi\lambda_1}$. It is a virtually abelian dihedral representation of the fundamental group $\Gamma_2 := \pi_1(\mathbb{P}^2 - \mathcal{Q})$ into $SL_2(\mathbb{C})$ whose image is not Zariski–dense.

The connection $\nabla_{\lambda_0,\lambda_1}$ is given in some (see Subsection 3.2.1) affine chart $\mathbb{C}^2_{x,y} \subset \mathbb{P}^2$ by:

$$\nabla_{\lambda_0,\lambda_1} = \mathrm{d} - \frac{1}{2(x^2 + y^2 + 1 - 2(xy + x + y))}(\lambda_0 A_0 + \lambda_1 A_1 + A_2) \ ,$$

where

$$A_0 := \begin{pmatrix} 2(x-1)y\mathrm{d}x + (x^2 + x(y-2) - y + 1)x\frac{\mathrm{d}y}{y} & 2(2x - y + 2)y\mathrm{d}x + (2x^2 + y(x - y + 3) - 2)x\frac{\mathrm{d}y}{y} \\ -2y^2\mathrm{d}x + (x + y - 1)x^2\frac{\mathrm{d}y}{y} & -2(x-1)y\mathrm{d}x - (x^2 + x(y-2) - y + 1)x\frac{\mathrm{d}y}{y} \end{pmatrix}$$

$$A_1 := \begin{pmatrix} (x^2 + (x-1)(y-1))y\frac{\mathrm{d}x}{x} + 2(x-1)x\mathrm{d}y & (x^2 + y(x - y + 3) - 2)y\frac{\mathrm{d}x}{x} + 2(2x - y + 2)x\mathrm{d}y \\ -(x + y - 1)y^2\frac{\mathrm{d}x}{x} - 2x^2\mathrm{d}y & -(x^2 + (x-1)(y-1))y\frac{\mathrm{d}x}{x} - 2(x-1)x\mathrm{d}y \end{pmatrix}$$

$$A_2 := \begin{pmatrix} -(x + y + 1)y\mathrm{d}x - (x^2 - x(y+2) - y + 1)x\frac{\mathrm{d}y}{y} & -2(x - y + 3)y\mathrm{d}x - (x^2 - 2y(x+1) + 1)x\frac{\mathrm{d}y}{y} \\ 0 & (x + y + 1)y\mathrm{d}x + (x^2 - x(y+2) - y + 1)x\frac{\mathrm{d}y}{y} \end{pmatrix} .$$

Remark 3.1.4. *Note that the existence, and uniqueness up to gauge transformation, of such a family of connections follows from Proposition 3.1.2 and the classical Riemann–Hilbert correspondence. The original part of this work resides in the fact that we give a constructive proof of this result; in particular this allows us to describe the associated algebraic Garnier solution.*

Since $\mathbb{P}^2$ is the symmetric product $\mathrm{Sym}^2(\mathbb{P}^1)$ one has a natural two–fold ramified cover $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{2:1} \mathbb{P}^2$ which pulls the quintic $\mathcal{Q}$ back onto the subset $D \subset X := \mathbb{P}^1 \times \mathbb{P}^1$ composed of the six lines $u_0, u_1 = 0, 1, \infty$ (for some pair (u_0, u_1) of projective coordinates on X) and of the diagonal Δ while ramifying over the latter (see Subsection 3.2.1 for more details). As we are aiming at dihedral monodromy, a natural idea to prove Theorem C is to define a family of rank one logarithmic flat connections over X with infinite monodromy around $D \setminus \Delta$ and to push it forward using π to get a family of such connections over $\mathbb{P}^2 - \mathcal{Q}$ with monodromy of (generically) infinite order around the three lines in the quintic and of projective order two at the conic $\mathcal{C}$. This is exactly what we will do in Section 3.2.

We also prove that this family of representations generically satisfies condition **(C2)**; more precisely we have the following result.

Theorem D. *The monodromy representation of the connections $\nabla_{\lambda_0,\lambda_1}$ introduced in Theorem C factors through an orbicurve if and only if there exists $(p, q) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $p\lambda_0 + q\lambda_1 = 0$, i.e if and only if $[\lambda_0 : \lambda_1] \in \mathbb{P}^1_{\mathbb{Q}}$.*

3.1.3 Isomonodromic deformations

By restriction to generic lines in $\mathbb{P}^2$, we obtain a family (parametrised by (λ_0, λ_1)) of isomonodromic deformations over the Riemann sphere with five pairwise distinct punctures $\mathbb{P}^1_x \setminus \{0, 1, t_1, t_2, \infty\}$, whose monodromy is given in Table 3.1, where x is a well chosen projective coordinate on $\mathbb{P}^1$ and t_1, t_2 are two independent variables (well defined up to double cover) corresponding to the intersection of the line with the conic $\mathcal{C}$. Since this family of connections is algebraic we get a two–parameter family of algebraic solutions of the isomonodromy equation associated with such deformations, namely the following Garnier system:

$$\begin{cases} \partial_{t_k} p_i &= -\partial_{q_i} H_k \\ \partial_{t_k} q_i &= \partial_{p_i} H_k \end{cases} \quad i, j = 1, 2 \quad , \tag{E3.1}$$

where $(p_i, q_i)_i$ are algebraic functions of t_1, t_2 and H_1, H_2 are explicit Hamiltonians given in Proposition 3.3.3 (see also [24, 46]). More precisely if one sets $S_q := q_1 + q_2$, $P_q := q_1 q_2$, $S_t := t_1 + t_2$ and $P_t := t_1 t_2$ one has the following relations:

$$\begin{cases} (\lambda_0 - 1)^2 \lambda_1^2 S_t = -F(S_q, P_q) \\ (\lambda_0 - 1)^2 P_t = -(\lambda_0 + \lambda_1 - 1)^2 P_q^2 \end{cases} ; \tag{E3.2}$$

where:

$$\begin{aligned} F(S_q, P_q) =& (\lambda_0 - \lambda_1 - 1)(\lambda_0 + \lambda_1 - 1)^3 P_q^2 \\ &+ (\lambda_0 - 1)^2 (\lambda_0 + \lambda_1 - 1)^2 (2P_q - 2P_q S_q + S_q^2 - 2Sq) \\ &+ (\lambda_0 - 1)^3 (\lambda_0 + 2\lambda_1 - 1) \, . \end{aligned}$$

These solutions generalise the two parameter family known for the Painlevé VI equation (see Subsection 3.3.1) and the complex surface associated with the graph of $(t_1, t_2) \mapsto (S_q, P_q)$ is rational.

$x = 0$	$x = 1$	$x = t_1$	$x = t_2$	$x = \infty$
$\begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}$	$\begin{pmatrix} -a_0 & 0 \\ 0 & -a_0^{-1} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a_0^2 \\ -a_0^{-2} & 0 \end{pmatrix}$	$\begin{pmatrix} a_0 a_1^{-1} & 0 \\ 0 & a_0^{-1} a_1 \end{pmatrix}$

Table 3.1: Monodromy on a generic line; here $a_j = \exp(-i\pi\lambda_j)$ for $j \in \{0, 1\}$.

3.1.4 Lotka–Volterra foliations

One last fact worth noting is that since ∇ is a flat $\mathfrak{sl}_2(\mathbb{C})$–connection on a trivial bundle, there exist three meromorphic one–forms α_0, α_1 and α_2 (given in Theorem C) such that

$$\nabla = \mathrm{d} + \Omega \ , \text{ where } \Omega := \begin{pmatrix} \alpha_1 & \alpha_0 \\ -\alpha_2 & -\alpha_1 \end{pmatrix} \text{ satisfies } \mathrm{d}\Omega = \Omega \wedge \Omega \, .$$

In particular, since $\mathrm{d}\alpha_2 \wedge \alpha_2 = 0$ one obtains a family of transversally projective degree two foliations over $\mathbb{P}^2$ (see [43] and Section 3.4; note that this family is therefore integrable in the Casale–Malgrange sense [12]) with invariant locus containing the quintic $\mathcal{Q}$. We show that these are conjugate to a family of Lotka-Volterra foliations over $\mathbb{C}^3$ [48, 49]; namely given three complex parameters (A, B, C), the codimension one foliation associated with the one–form over $\mathbb{C}^3$, with coordinates (x, y, t):

$$\omega_0 := (yV_t - tV_y)\mathrm{d}x + (tV_x - xV_t)\mathrm{d}y + (xV_y - yV_x)\mathrm{d}t \ ,$$

where:

$$V_x := x(Cy + t), \qquad V_y := y(At + x) \qquad \text{and} \qquad Vt := t(Bx + y) \ .$$

Theorem E. *The foliation defined by the meromorphic one–form α_2 is equal to the foliation over $\mathbb{P}^2$ associated with a Lotka–Volterra system with parameters*

$$(A, B, C) = \left(\frac{\lambda_1}{\lambda_0}, \frac{-\lambda_0}{\lambda_0 + \lambda_1}, \frac{-(\lambda_0 + \lambda_1)}{\lambda_1} \right) \quad .$$

Conversely, any degree two foliation whose invariant locus contains the quintic $\mathcal{Q}$ is equal to one of the above form.

One can see from Theorem E that this family of foliations is governed by the parameter λ_0/λ_1; there exists a one–parameter family of connections corresponding to any given foliation (see also Subsection 4.4 in [43]). We then prove that this gives an example of a family of foliations with algebraic invariant curves of arbitrarily high degree (see also [41]).

3.2 Proof of Theorem C

In this section we concern ourselves with setting up a particular fibre bundle over the projective plane $\mathbb{P}^2$, and then endowing it with a family of logarithmic flat connections satisfying the conditions of Theorem C.

3.2.1 A rank two fibre bundle

Start by considering the complex manifold $X := \mathbb{P}^1 \times \mathbb{P}^1$ and define the following involution:

$$\begin{aligned}\tilde{\eta} : X &\to X \\ (u_0, u_1) &\mapsto (u_1, u_0) \ .\end{aligned}$$

The action of $\tilde{\eta}$ gives us a two–fold ramified cover of $\mathbb{P}^1 \times \mathbb{P}^1$ over the projective plane $\mathbb{P}^2$, i.e the fibres of the morphism

$$\begin{aligned}\mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^2 \\ ([u_0^0 : u_0^1], [u_1^0 : u_1^1]) &\mapsto [s : p : t] := [(u_0^0 u_1^1 + u_0^1 u_1^0) : u_0^0 u_1^0 : u_0^1 u_1^1] \,.\end{aligned}$$

are the orbits under $\bar{\eta}$. However, for the purpose of this chapter, we will compose this mapping with the linear projective transformation of $\mathbb{P}^2$ given by:

$$[s : p : t] \mapsto [p + t - s : p : t] \ .$$

This means that we will now work in the homogeneous coordinates $x := p + t - s$, $y := p$ and t. We get a two–fold ramified cover $\pi : X \to \mathbb{P}^2$ that ramifies along the diagonal $\Delta := (u_0 = u_1) \subset X$ and sends it onto the conic:

$$(\mathcal{C}) \qquad x^2 + y^2 + t^2 = 2(xy + xt + yt) \ .$$

Now consider the rank two fibre bundle E over the projective plane associated with the locally free sheaf

$$\mathcal{E} := \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \,.$$

Let e_+ be some global nonvanishing holomorphic section of E (corresponding to the $\mathcal{O}_{\mathbb{P}^2}$ part of the above decomposition) and e_- be some global meromorphic section linearly independent from e_+ (and so corresponding to $\mathcal{O}_{\mathbb{P}^2}(-1)$) with associated (zeroes and poles) divisor equal to $-L_\infty$, where L_∞ is the line "at infinity" ($t = 0$).

Let us ask ourselves the following question: what does the pullback sheaf $\mathcal{F} := \pi^*\mathcal{E}$ look like ? For any open set $U \subset X$ we have $\mathcal{F}(U) = \mathcal{E}(\pi(U))$, which implies that $\mathcal{F}$ is a rank two locally free sheaf inducing a rank two fibre bundle $F \to X$ with two global section: one nonvanishing holomorphic $e_1 := \pi^*e_+$ and one meromorphic $e_2 := \pi^*e_-$. Since π does not ramify over $L_\infty^0 := (u_0 = \infty)$ nor $L_\infty^1 := (u_1 = \infty)$, e_2 has associated divisor $-(L_\infty^0 + L_\infty^1)$; thus:

$$\mathcal{F} \cong \mathcal{O}_X \oplus \mathcal{O}_X(-1, -1) \,.$$

To better understand the bundle F, start by considering the rank two trivial bundle $E_0 := \mathbb{C}^2 \times X \to X$ over X; it has two independent (constant) holomorphic global sections

$f_1 \equiv (1,0)$ and $f_2 \equiv (0,1)$. Define the following involution:

$$E_0 \to E_0$$
$$(u_0, u_1, (Z_1, Z_2)_{e_1,e_2}) \mapsto (u_1, u_0, (Z_1, -Z_2)_{e_1,e_2}) .$$

First of all, note that its action on the base coincides with that of the involution $\bar{\eta}$. One can then identify two global invariant sections of the bundle E_0:

- f_1, which is holomorphic;
- $\hat{f}_2 := b \cdot f_2$, where b is the global meromorphic function $(u_0, u_1) \mapsto u_0 - u_1$.

The local expression $b \cdot f_2$ defines a global meromorphic section with associated divisor $\Delta - (L^0_\infty + L^1_\infty)$. The $\mathcal{O}_X$–module spanned by the sections f_1 and $\hat{f}_2$ is isomorphic to the rank two locally free module $\mathcal{F}$ (by mapping f_1 to e_1 and $\hat{f}_2$ to e_2) and as such defines a rank two vector bundle over X isomorphic to F. More precisely, one goes (locally) from E_0 to F using the following transformation (which is trivial on the base):

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}_{e_1,e_2} \mapsto \begin{pmatrix} Z_1 \\ \frac{Z_2}{u_0-u_1} \end{pmatrix}_{e_1,\tilde{e_2}} .$$

3.2.2 A rank one projective bundle

By quotient on the fibres ($\mathbb{P}(\mathbb{C}^2) = \mathbb{P}^1_{\mathbb{C}}$), one can associate to both vector bundles E_0 and E rank one projective bundles $\mathbb{P}^1 \times X$ and $\mathbb{P}(E)$. We can describe the action of η on the former as follows:

$$\eta : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$
$$(u_0, u_1, [Z_1 : Z_2]) \mapsto (u_1, u_0, [Z_2 : Z_1]) \quad ;$$

or, in the affine chart "$z = \frac{Z_1}{Z_2}$", $(u_0, u_1, z) \mapsto (u_1, u_0, 1/z)$.

One goes from $\mathbb{P}^1 \times X$ to $\mathbb{P}(E)$ through the following invariant rational functions:

$$\begin{cases} s = u_0 + u_1 \\ p = u_0 u_1 \\ w := (u_0 - u_1)\dfrac{z+1}{z-1} \end{cases} ;$$

here (s, p) gives us local coordinates over the base and w does the same in the fibres. We will use this projective point of view throughout this chapter as it allows for easier computations in the long run. It will also allow us to define an interesting family of Lotka–Volterra foliations in Section 3.4.

3.2.3 Logarithmic flat connections

Start by endowing the trivial rank two vector bundle $E_0 \to X$ with the following logarithmic flat connection :

$$\nabla_0 := \mathrm{d} + \frac{1}{2}\begin{pmatrix} \omega_0 & 0 \\ 0 & -\omega_0 \end{pmatrix} ,$$

where u_0, u_1 are projective coordinates on the base X and

$$\omega_0 := \lambda_0 \left(\frac{\mathrm{d}u_0}{u_0} - \frac{\mathrm{d}u_1}{u_1} \right) + \lambda_1 \left(\frac{\mathrm{d}u_0}{u_0 - 1} - \frac{\mathrm{d}u_1}{u_1 - 1} \right) ,$$

with $(\lambda_0, \lambda_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$. This connection has singular locus equal to six lines in X (if $\lambda_0\lambda_1(\lambda_0 + \lambda_1) \neq 0$) and naturally gives rise to a Riccati foliation, defined by the following one–form over $\mathbb{P}(E_0) = X \times \mathbb{P}^1$:

$$\mathfrak{R}(\nabla_0) := \mathrm{d}z + \omega_0 z \text{ where } z \text{ is a projective coordinate on the fibres.}$$

Moreover one easily checks that $\frac{u_0 - u_1}{z}\mathfrak{R}(\nabla_0)$ is an η–invariant logarithmic one–form over $\mathbb{P}(F)$, associated with some connection ∇_1 in the following sense: if one has, in some local chart

$$(u_0 - u_1)\mathfrak{R}(\nabla_0) = \mathrm{d}z + \alpha_2 z^2 + 2\alpha_1 z + \alpha_0 ,$$

with $\alpha_0, \alpha_1, \alpha_2$ meromorphic one–forms (remark that here α_2 and α_0 are zero), then one can set (in the same local chart)

$$\nabla_1 := \mathrm{d} + \begin{pmatrix} \alpha_1 & \alpha_0 \\ -\alpha_2 & -\alpha_1 \end{pmatrix} .$$

Since the associated Riccati form $\mathfrak{R}(\nabla_1) := (u_0 - u_1)\mathfrak{R}(\nabla_0)$ is η–invariant one can use ∇_1 to get a logarithmic connection ∇_2 on E with poles along $(y = 0)$, $(x = 0)$, L_∞ and $\mathcal{C}$, the latter coming from the fact that π ramifies there. More precisely, in the affine chart described in Subsection 3.2.2 one has:

$$\mathfrak{R}(\nabla_2) = \mathrm{d}w + \frac{1}{2(x^2 + y^2 + 1 - 2(xy + x + y))}\left(g(\lambda_0, \lambda_1, w, x, y)\frac{\mathrm{d}x}{x} + g(\lambda_1, \lambda_0, w, y, x)\frac{\mathrm{d}y}{y} \right) ,$$

where

$$\begin{aligned} g(\lambda_0, \lambda_1, w, x, y) = & - ((2\lambda_0 + \lambda_1)x + \lambda_1(y - 1))w^2 + 2(y - x + 1)xw \\ & + (2\lambda_0 + \lambda_1)x^3 - ((4\lambda_0 + \lambda_1)(y + 1) + 2\lambda_1)x^2 \\ & - ((-2\lambda_0 + \lambda_1)y^2 + 2(2\lambda_0 + \lambda_1)y - (2\lambda_0 + 3\lambda_1))x \\ & + \lambda_1(y^3 - 3y^2 + 3y - 1) . \end{aligned}$$

3.2.4 Trivialisations

We wish to turn ∇_2 into a connection on the trivial bundle $\mathbb{C}^2 \times \mathbb{P}^2$; this can be done simply by blowing up the pole of any global meromorphic section of $\mathbb{P}(E)$ then contracting a suitable divisor, however we want to do so without disturbing the logarithmic nature of the connection ∇_2.

Lemma 3.2.1. *There exists a birational mapping $\Phi : \mathbb{P}(E) \to \mathbb{P}^1 \times \mathbb{P}^2$ conjugating $\mathfrak{R}(\nabla_2)$ to some Riccati one–form, that is associated with a logarithmic flat connection ∇ over the trivial bundle $\mathbb{C}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$.*

Proof. First remark that we have the following local expression along $(y = 0)$:

$$\mathfrak{R}(\nabla_2)_{|(y=0)} = \mathrm{d}w + f(x)(w + x - 1)(w - x + 1)\frac{\mathrm{d}y}{y} .$$

This tells us that the codimension one foliation associated with the one–form $\mathfrak{R}(\nabla_2)$ has two singular points on each fibre above $(y = 0)$, namely at $w = \pm(x - 1)$. So in order to get a birationnal map $\mathbb{P}(E) \to \mathbb{P}^1 \times \mathbb{P}^2$ one can proceed as follows:

- move one of the aforementioned singular loci (e.g $(w = x - 1) \cap (y = 0)$) at $(w = y = 0)$;
- blow up $(y = 0) \cap (w = \infty)$ then contract the strict transform of the fibre at $(y = 0)$ on $(w = y = 0)$. This latest step is achieved (in our usual affine chart) through the birational map $(x, y, w) \mapsto (x, y, w/y)$.

Explicitly in our local chart, the mapping Φ is given by

$$\Phi(w, x, y) = (y(w - x + 1), x, y) .$$

This means that we are blowing up (inside the total space) a line in each fibre over $(y = 0)$ then contracting the strict transforms of said fibres thus resolving the singularities of the global meromorphic section e_- described in the proof of Theorem C; this shows that our mapping does indeed end in a trivial bundle and since we took care of contracting divisors only on points of the singular locus of the foliation associated with $\mathfrak{R}(\nabla_2)$ we get a logarithmic flat connection over $\mathbb{C}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$. □

In the end, one gets a connection $\nabla = \nabla_{\lambda_0,\lambda_1}$ on the trivial bundle $\mathbb{C}^2 \times \mathbb{P}^2$ that almost satisfies condition (i) in Theorem C, the only thing left to check being whether or not it is a $\mathfrak{sl}_2(\mathbb{C})$–connection. Explicitly, the Riccati form associated with ∇ is given by (in the affine chart $(t = 1)$):

$$\mathfrak{R}(\nabla) = \mathrm{d}w - \frac{1}{2(x^2 + y^2 + 1 - 2(xy + x + y))}\left(f_1(w, x, y)\frac{y\mathrm{d}x}{x} - f_2(w, x, y)\frac{x\mathrm{d}y}{y}\right)$$

where

$$\begin{aligned} f_1(x,y) =& ((2\lambda_0+\lambda_1)x+\lambda_1(y-1))yw^2 \\ &+2((2\lambda_0+\lambda_1-1)x^2+((\lambda_1+1)y-(2\lambda_0+2\lambda_1-1))x-\lambda_1(y-1))w \\ &+2(2\lambda_0+\lambda_1-1)x^2+((-2\lambda_0+\lambda_1+2)y+2(2\lambda_0-3))x \\ &+\lambda_1(-y^2+3y-2) \end{aligned}$$

and

$$\begin{aligned} f_2(x,y) =& (\lambda_0(x-1)+(\lambda_0+2\lambda_1)y)xw^2 \\ &+2((\lambda_0-1)(x^2+1)+((\lambda_0+2\lambda_1+1)y-2(\lambda_0-1))x-(\lambda_0+2\lambda_1-1)y)w \\ &+2(\lambda_0-1)(x^2-1)+(\lambda_0+4\lambda_1+2)yx-(\lambda_0+2\lambda_1)y^2+(3\lambda_0+4\lambda_1-2)y\ . \end{aligned}$$

Note that our birational transformation has "broken" the symmetry between the two components f_1 and f_2.

We can explicitly compute the residues of the connection $\nabla=\nabla_{\lambda_0,\lambda_1}$ and so check that it is indeed a $\mathfrak{sl}_2(\mathbb{C})$–connection (see Table 3.2); note that the eigenvalues at $(y=0)$ have been slightly modified because we moved the singular points of the associated foliation.

Divisor	Residue	Eigenvalues
$y=0$	$\begin{pmatrix} -\frac{1}{2}\lambda_0+\frac{1}{2} & \frac{2(\lambda_0-1)}{y-x+1} \\ 0 & \frac{1}{2}\lambda_0-\frac{1}{2} \end{pmatrix}$	$\pm\dfrac{\lambda_0-1}{2}$
$x=0$	$\begin{pmatrix} -\frac{\lambda_1(y-x+1)}{2(y-x-1)} & \frac{2\lambda_1}{y-x-1} \\ -\frac{\lambda_1(y-x)}{2(y-x-1)} & \frac{\lambda_1(y-x+1)}{2(y-x-1)} \end{pmatrix}$	$\pm\dfrac{\lambda_1}{2}$
$\mathcal{C}$	$\begin{pmatrix} \frac{(2\lambda_0+2\lambda_1-1)(y-x+1)-4\lambda_0+2}{4(y-x-1)} & -\frac{2((\lambda_0+\lambda_1-1)(y-x+1)-2\lambda_0+2)}{(y-x+1)(y-x-1)} \\ \frac{(\lambda_0+\lambda_1)(y-x+1)(y-x-1)}{8(y-x-1)} & -\frac{(2\lambda_0+2\lambda_1-1)(y-x+1)-4\lambda_0+2}{4(y-x-1)} \end{pmatrix}$	$\pm\dfrac{1}{4}$
L_∞ $(X=x/y)$	$\begin{pmatrix} -\frac{1}{2}\lambda_0-\frac{1}{2}\lambda_1 & 0 \\ \frac{\lambda_0+\lambda_1}{2(X-1)} & \frac{1}{2}\lambda_0+\frac{1}{2}\lambda_1 \end{pmatrix}$	$\pm\dfrac{\lambda_0+\lambda_1}{2}$

Table 3.2: Residues for ∇.

3.2.5 Monodromy representation

To conclude the proof of Theorem C one needs to compute the monodromy representation of the connection ∇ and see that it is, as announced, a dihedral representation of Γ_2 into $SL_2(\mathbb{C})$.

For $j = 0, 1$ set $a_j := e^{-i\pi\lambda_j}$; the monodromy associated with the connection ∇_0 is as follows:

- around $u_0 = j$ (resp. $u_1 = j$), $j = 0, 1$, it is the multiplication by a_j (resp. a_j^{-1});
- around $u_0 = \infty$ (resp. $u_1 = \infty$), it is the multiplication by $a_0^{-1}a_1^{-1}$ (resp. $a_0 a_1$).

This is a complete description since the fundamental group of the projective line minus six lines is isomorphic to $\mathbf{F}_2 \times \mathbf{F}_2$ and is generated by loops going around $x, y = 0, 1$ once.

The monodromy of the connection ∇_2 comes directly from that of ∇_0 around the three lines in its singular locus; more precisely we can explicitly compute (up to conjugacy) its local monodromy around:

- $(y = 0)$:
$$\begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix} ;$$
- $(x = 0)$:
$$\begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} ;$$
- and L_∞:
$$\begin{pmatrix} (a_0 a_1)^{-1} & 0 \\ 0 & a_0 a_1 \end{pmatrix} .$$

However the monodromy of ∇_2 around the conic $\mathcal{C}$ comes solely from the ramification of the cover π. More precisely since any path linking $(u_0, u_1) \in X$ to (u_1, u_0) pushes back as a loop on the quotient $\mathbb{P}^2 = \pi(X)$ and since any local solution z of ∇_0 satisfies $z(u_1, u_0) = \dfrac{1}{z(u_0, u_1)}$ the monodromy group of ∇_2 must contain the following matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Proposition 3.2.2. *The monodromy group of the connection ∇ is the subgroup of the infinite dihedral group*

$$\mathbf{D}_\infty := \left\{ \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} , \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}^* \right\} \leq SL_2(\mathbb{C})$$

generated by the following three matrices:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -e^{-i\pi\lambda_0} & 0 \\ 0 & -e^{i\pi\lambda_0} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} e^{-i\pi\lambda_1} & 0 \\ 0 & e^{i\pi\lambda_1} \end{pmatrix} .$$

Proof. We know from Degtyarev's list [19] that the fundamental group of the complement of the singular locus of ∇ in $\mathbb{P}^2$ has the following presentation:

$$\Gamma_2 = \langle a, b, c \,|\, (ab)^2(ba)^{-2} = (ac)^2(ca)^{-2} = [b, c] = 1 \rangle \ ;$$

and that we can take a to be a loop whose lift is some path in X joining (x, y) and (y, x) (for generic $(x, y) \in X$) and b (resp. c) to be a loop going around $(y = 0)$ (resp. $(x = 0)$) once (see Figure 3.1). If we choose a set of local coordinates in which the monodromy matrices of both b and c are diagonal (this is possible because the two loops commute) then the monodromy of a only comes from the cover π and is equal to:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

In conclusion, the monodromy representation is given by the following matrices:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -e^{-i\pi\lambda_0} & 0 \\ 0 & -e^{i\pi\lambda_0} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^{-i\pi\lambda_1} & 0 \\ 0 & e^{i\pi\lambda_1} \end{pmatrix} ,$$

which are elements of $\mathbf{D}_\infty$. □

3.3 Algebraic Garnier solutions

In this section we show that the connection ∇ induces an isomonodromic deformation over the four and five punctured spheres. Furthermore we give rational parametrisations of the associated algebraic Painlevé VI and Garnier solutions and a description of the associated monodromy representation.

3.3.1 Painlevé VI solutions

It is well known [34, 36] that isomonodromic deformations of rank two $\mathfrak{sl}_2(\mathbb{C})$–connections over the four punctured sphere correspond to solutions of the sixth Painlevé equation,

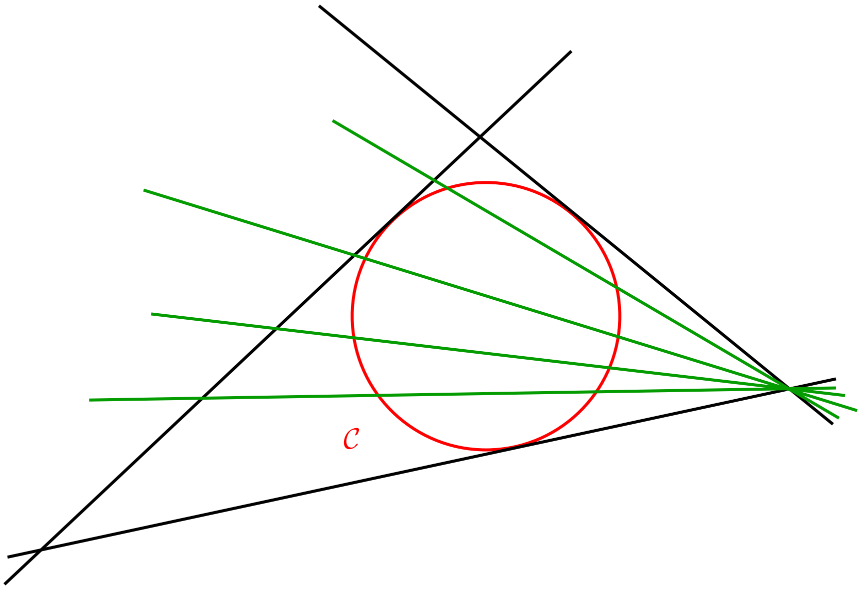

Figure 3.2: Special lines.

namely the following order two nonlinear differential equation:

$$\begin{aligned}\frac{d^2q}{du^2} =&\frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-u}\right)\left(\frac{\mathrm{d}q}{du}\right)^2\\&-\left(\frac{1}{u}+\frac{1}{u-1}+\frac{1}{q-u}\right)\frac{\mathrm{d}q}{du}\\&+\frac{q(q-1)(q-u)}{u^2(u-1)^2}\left(\alpha+\beta\frac{u}{q^2}+\gamma\frac{u-1}{(q-1)^2}+\delta\frac{u(u-1)}{(q-u)^2}\right) ,\end{aligned}$$

where α,β,γ and δ are complex–valued parameters.

Let us look at the connection induced by ∇ on the family of lines going through $P_0 := (x=0)\cap L_\infty$ (see Figure 3.2) that are neither $(x=0)$ nor the line at infinity; these are the lines of the form $(y=c)$ in the affine chart (x,y) from Subsection 3.2.1. According to Subsection 3.2.4, this corresponds to studying the isomonodromic deformation given by the following Riccati forms, for generic y:

$$\mathfrak{R}(\nabla_y) := \mathrm{d}w - \frac{y}{2x(x^2+y^2+1-2(xy+x+y))}f_y(x,w)\mathrm{d}x ,$$

where

$$\begin{aligned}f_y(x,w) =&(\lambda_0(x-1)+(\lambda_0+2\lambda_1)y)yw^2\\&+2((\lambda_0-1)(x^2+1)+((\lambda_0+2\lambda_1+1)y-2(\lambda_0-1))x-(\lambda_0+2\lambda_1-1)y)w\\&+2(\lambda_0-1)(x^2-1)+(\lambda_0+4\lambda_1+2)yx-(\lambda_0+2\lambda_1)y^2+(3\lambda_0+4\lambda_1-2)y .\end{aligned}$$

From this isomonodromic deformation we produce algebraic solutions of the Painlevé VI

Pole	Residue	Eigenvalues
$x=0$	$W_0 := \begin{pmatrix} -\frac{\lambda_1(z^2+1)}{2\,(z^2-1)} & \frac{2\,\lambda_1}{z^2-1} \\ -\frac{\lambda_1 z^2}{2\,(z^2-1)} & \frac{\lambda_1(z^2+1)}{2\,(z^2-1)} \end{pmatrix}$	$\pm\frac{\lambda_1}{2}$
$x=1$	$W_1 := \begin{pmatrix} \frac{(2\,\lambda_0+2\,\lambda_1-1)z+2\,\lambda_0-1}{4\,(z+1)} & \frac{(\lambda_0+\lambda_1-1)z+\lambda_0-1}{z^2+z} \\ -\frac{(\lambda_0+\lambda_1)z^2+\lambda_0 z}{4\,(z+1)} & -\frac{(2\,\lambda_0+2\,\lambda_1-1)z+2\,\lambda_0-1}{4\,(z+1)} \end{pmatrix}$	$\pm\frac{1}{4}$
$x=t(z)$	$W_2 := \begin{pmatrix} \frac{(2\,\lambda_0+2\,\lambda_1-1)z-2\,\lambda_0+1}{4\,(z-1)} & -\frac{(\lambda_0+\lambda_1-1)z-\lambda_0+1}{z^2-z} \\ \frac{(\lambda_0+\lambda_1)z^2-\lambda_0 z}{4\,(z-1)} & -\frac{(2\,\lambda_0+2\,\lambda_1-1)z-2\,\lambda_0+1}{4\,(z-1)} \end{pmatrix}$	$\pm\frac{1}{4}$
$x=\infty$	$W := \begin{pmatrix} -\lambda_0-\frac{1}{2}\,\lambda_1+\frac{1}{2} & 0 \\ 0 & \lambda_0+\frac{1}{2}\,\lambda_1-\frac{1}{2} \end{pmatrix}$	$\pm\frac{2\lambda_0+\lambda_1-1}{2}$

Table 3.3: Residues for ∇_y.

equation by adapting part of a paper by Hitchin [34].

Proposition 3.3.1. *The family of algebraic solutions of the Painlevé VI equation associated with the connections* $(\nabla_{\lambda_0,\lambda_1})_{\lambda_0,\lambda_1}$ *is given by the functions*

$$q(u) = -\frac{\lambda_1}{2\lambda_0+\lambda_1}\sqrt{u}$$

and the parameters:

$$\alpha = \frac{(2\lambda_0+\lambda_1)^2}{2}\ ,\beta = -\frac{\lambda_1^2}{2}\ ,\gamma = 1/8 \text{ and } \delta = 3/8\ .$$

Proof. Let z be a parameter such that $z^2 = y$; then ∇_y has poles at $x = (z\pm 1)^2$, $x = 0$ and $x = \infty$. Up to Möbius transformation, one can assume that these are in fact located at $s \in \{0, 1, u(z), \infty\}$, with:

$$u(z) = \frac{z^2-2z+1}{z^2+2z+1} = \frac{(z-1)^2}{(z+1)^2}\ .$$

It is then possible to compute the relevant data associated with this family of connections (see Table 3.3).

Let us now set

$$H := \frac{W_0}{x} + \frac{W_1}{x-1} + \frac{W_2}{x-t}\ ,$$

where the W_i are the residues from Table 3.3; then since $-W$ is diagonal and equal to the sum $W_0 + W_1 + W_2$, its lower left coefficient is a degree one polynomial in x, whose root can be explicitly computed as a rational function of z:

$$q(z) := -\frac{\lambda_1}{2\lambda_0 + \lambda_1} \frac{z-1}{z+1} \,,$$

or as an algebraic function of u:

$$q(u) = -\frac{\lambda_1}{2\lambda_0 + \lambda_1} \sqrt{u} \,.$$

One can then check that this function $u \mapsto q(u)$ is indeed a solution of the sixth Painlevé equation for the announced choice of parameters.

□

3.3.2 Restriction to generic lines

Let us now consider the connection induced by ∇ on generic lines in $\mathbb{P}^2$, such a line being given in our usual affine chart by an equation of the form $y = \alpha x + \beta$. We thus obtain an isomonodromic deformation $(\nabla_{\alpha,\beta})_{\alpha,\beta}$ over the five punctured sphere; more precisely if one chooses a parameter z such that $z^2 = \beta(1-\alpha) + \alpha$ then one gets (after Möbius transformation) a family of logarithmic flat connections over $\mathbb{P}^1 \setminus \{0, 1, t_1, t_2, \infty\}$, where:

$$t_1 = -\frac{\alpha(z+1)^2}{(\alpha-1)(\alpha-z^2)} \quad \text{and} \quad t_2 = -\frac{\alpha(z-1)^2}{(\alpha-1)(\alpha-z^2)} \,.$$

The associated Riccati forms are given by:

$$\mathfrak{R}(\nabla_{\alpha,\beta}) = \mathrm{d}w + \frac{a_2(x)w^2 + a_1(x)w + a_0(x)}{2x(x-1)(x-t_1)(x-t_2)} \mathrm{d}x$$

where:

$$\begin{aligned} \frac{a_2(x)}{\alpha(x-1)(z^2-\alpha)} =& (\lambda_0 + \lambda_1)(\alpha^2 - (z^2+1)\alpha + z^2)x^2 \\ &+ (-\lambda_1\alpha^2 + (\lambda_0(z^2+1) + 2\lambda_1)\alpha - (2\lambda_0 + \lambda_1)z^2)x \\ &+ \lambda_1(z^2-1)\alpha \end{aligned}$$

$$
\begin{aligned}
\frac{a_1(x)}{2} =&(\lambda_0+\lambda_1)(\alpha^4-2(z^2+1)\alpha^3+(z^4+4z^2+1)\alpha^2-2(z^4+z^2)\alpha+z^4)x^3\\
&+[-(2\lambda_0+3\lambda_1-1)\alpha^4\\
&\quad+((4\lambda_0+4\lambda_1-1)z^2+4\lambda_0+6\lambda_1-1)\alpha^3\\
&\quad-((2\lambda_0+\lambda_1)z^4+2(4\lambda_0+\lambda_1-1)z^2+(2\lambda_0+3\lambda_1))\alpha^2\\
&\quad+((4\lambda_0+2\lambda_1-1)z^4+(4\lambda_0+4\lambda_1-1)z^2)\alpha\\
&\quad-(2\lambda_0+\lambda_1+1)z^4]x^2\\
&+[2\lambda_1 a^4-((2\lambda_0-1)z^2+(2\lambda_0+6\lambda_1-1))\alpha^3\\
&\quad+((\lambda_0-\lambda_1)z^4+2(3\lambda_0+2\lambda_1-1)z^2+\lambda_0+3\lambda_1)\alpha^2\\
&\quad+((2\lambda_0-1)z^4+(2\lambda_0+2\lambda_1-1)z^2)\alpha]x\\
&+\lambda_1(2(1-z^2)\alpha+z^4-1)\alpha^2
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{a_0(x)}{4\alpha(\alpha-1)} =&(\lambda_0+\lambda_1-1)(1-\alpha)(z^2-\alpha)x^2\\
&+(((\lambda_0-1)(\alpha-2)-\lambda_1)z^2-\lambda_1\alpha^2+(\lambda_0+2\lambda_1-1)\alpha)x\\
&+\lambda_1\alpha(z^2-1)\ .
\end{aligned}
$$

Using the explicit formulas given in Subsection 3.2.4, we can explicitly compute the spectral data associated with these connections (see Table 3.4). To mirror what we did in Subsection 3.3.1, let us assume (up to a change of basis) that the residue at infinity M is diagonal and set:

$$\hat{H} := \frac{M_0}{x}+\frac{M_1}{x-1}+\frac{M_{t_1}}{x-t_1}+\frac{M_{t_2}}{x-t_2}\ ;$$

then since M does not depend on x, the lower left coefficient of $\hat{H}$ must be a degree two polynomial in x, say:

$$\hat{H}_{2,1} = \frac{c(t_1,t_2)(x^2-S_q(t_1,t_2)x+P_q(t_1,t_2))}{x(x-1)(x-t_1)(x-t_2)}\,, \tag{E3.3}$$

where $S_q := q_1+q_2$ and $P_q := q_1q_2$, with q_1, q_2 some algebraic functions of (t_1,t_2).

3.3.3 Rational parametrisations

First remark that one can rewrite (E3.3) as follows:

$$x(x-1)(x^2-S_tx+P_t)\hat{H}_{2,1} = c(t_1,t_2)(x^2-S_qx+P_q)\,,$$

where $S_t = t_1+t_2$ and $P_t = t_1t_2$ are the elementary symmetric polynomials in (t_1,t_2).

Lemma 3.3.2. *The parameters (α, z) introduced in Subsection 3.3.2 give a rational map-*

Pole	Residue	Eigenvalues
$x=0$	$M_0 := \begin{pmatrix} -\frac{\lambda_1(z^2-2\alpha+1)}{2\,(z^2-1)} & -\frac{2\lambda_1(\alpha-1)}{z^2-1} \\ -\frac{\lambda_1(z^2-\alpha)}{2\,(z^2-1)} & \frac{\lambda_1(z^2-2\alpha+1)}{2\,(z^2-1)} \end{pmatrix}$	$\pm\dfrac{\lambda_1}{2}$
$x=1$	$M_1 := \begin{pmatrix} -\frac{1}{2}\,\lambda_0+\frac{1}{2} & -\frac{2\alpha(\alpha-1)(1-\lambda_0)}{\alpha^2-z^2} \\ 0 & \frac{1}{2}\,\lambda_0-\frac{1}{2} \end{pmatrix}$	$\pm\dfrac{1}{2}(\lambda_0-1)$
$x=t_1$	$M_{t_1} := \begin{pmatrix} \frac{(2\lambda_0-1)(z+1)+2\lambda_1(\alpha+z)}{4\,(z+1)} & -\frac{(\alpha-1)(\lambda_0+\lambda_1\alpha-1+(\lambda_0+\lambda_1-1)z)}{(\alpha+1)z+z^2+\alpha} \\ \frac{\lambda_0(z^2+(1+\alpha)z+\alpha)+\lambda_1(z+\alpha)^2}{4\,((\alpha-1)z+\alpha-1)} & -\frac{(2\lambda_0-1)(z+1)+2\lambda_1(\alpha+z)}{4\,(z+1)} \end{pmatrix}$	$\pm\dfrac{1}{4}$
$x=t_2$	$M_{t_2} := \begin{pmatrix} -\frac{(2\lambda_0-1)(1-z)+2\lambda_1(\alpha-t)}{4\,(z-1)} & \frac{(\alpha-1)(\lambda_0+\alpha\lambda_1-1-(\lambda_0+\lambda_1-1)z)}{(\alpha+1)z-z^2-\alpha} \\ -\frac{\lambda_0(z^2+(1+\alpha)z+\alpha)+\lambda_1(z+\alpha)^2}{4\,((\alpha-1)z-\alpha+1)} & \frac{(2\lambda_0-1)(1-z)+2\lambda_1(\alpha-z)}{4\,(z-1)} \end{pmatrix}$	$\pm\dfrac{1}{4}$
$x=\infty$	$M := \begin{pmatrix} -\frac{1}{2}\,\lambda_0-\frac{1}{2}\,\lambda_1 & 0 \\ -\frac{(\lambda_0+\lambda_1)\alpha}{2\,(\alpha-1)} & \frac{1}{2}\,\lambda_0+\frac{1}{2}\,\lambda_1 \end{pmatrix}$	$\pm\dfrac{1}{2}(\lambda_0+\lambda_1)$

Table 3.4: Residues for $\nabla_{\alpha,\beta}$.

ping $(\mathbb{P}^1)^2 \to (\mathbb{P}^1)^4$ *giving explicit expressions of* (t_1, t_2, S_q, P_q)*, namely:*

$$
\begin{aligned}
t_1 &= -\frac{\alpha(z+1)^2}{(\alpha-1)(\alpha-z^2)}\,,\\
t_2 &= -\frac{\alpha(z-1)^2}{(\alpha-1)(\alpha-z^2)}\,,\\
S_q &= \frac{\lambda_0(\alpha^2-2\alpha+z^2)-\lambda_1(1+z^2+2\alpha)\alpha+\alpha(2-\alpha)-z^2}{(\lambda_0+\lambda_1-1)(\alpha-z^2)(\alpha-1)}\,,\\
P_q &= \frac{(\lambda_0-1)(z-1)(z+1)\alpha}{(\lambda_0+\lambda_1-1)(\alpha-z^2)(\alpha-1)}\,.
\end{aligned}
$$

Proof. Using Gröbner bases to eliminate the variable x one obtains a system of equations of the following form:

$$
\begin{cases}
(\lambda_0-1)^2\lambda_1^2 S_t = -F(S_q, P_q)\\
(\lambda_0-1)^2 P_t = -(\lambda_0+\lambda_1-1)^2 P_q^2
\end{cases}; \tag{E3.4}
$$

where:

$$
\begin{aligned}
F(S_q, P_q) =&(\lambda_0-\lambda_1-1)(\lambda_0+\lambda_1-1)^3 P_q^2\\
&+(\lambda_0-1)^2(\lambda_0+\lambda_1-1)^2(2P_q-2P_qS_q+S_q^2-2Sq)\\
&+(\lambda_0-1)^3(\lambda_0+2\lambda_1-1)\,.
\end{aligned}
$$

The discriminant Δ_t of this system vanishes along 2 pairs of parallel lines in $\mathbb{P}^1_{S_q} \times \mathbb{P}^1_{P_q}$; namely:

$$
(\Delta_t = 0) = (\alpha = 0) \cup (x' = 0) \cup (\alpha = \infty) \cup (x' = \infty) \subset \mathbb{P}^1_\alpha \times \mathbb{P}^1_{x'}
$$

for some projective coordinate x' such that $z^2 = \alpha x'$. This explicit description of the two–fold ramified cover given by z allows us to parametrize (S_q, P_q) as rational functions of (α, z), hence concluding the proof. □

We can now prove that we have indeed constructed a family of algebraic solutions for a Garnier system. More precisely, consider the following Hamiltonian system:

$$
\begin{cases}
\partial_{t_k}\mathbf{p}_i &= -\partial_{\mathbf{q}_i}H_k \quad i,k=1,2\\
\partial_{t_k}\mathbf{q}_i &= \partial_{\mathbf{p}_i}H_k \quad i,k=1,2
\end{cases}, \tag{E3.5}
$$

where:

$$
H_k := (-1)^k\frac{2H(t_k, t_{3-k}, \mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2)+H(t_k, t_{3-k}, \mathbf{p}_2, \mathbf{p}_1, \mathbf{q}_2, \mathbf{q}_1)}{2(\mathbf{q}_1-\mathbf{q}_2)(t_1-t_2)(t_k-1)t_k}
$$

with:

$$\begin{aligned}\frac{H(t_1,t_2,\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}_1,\mathbf{q}_2)}{\mathbf{p}_1\mathbf{q}_1(\mathbf{q}_2-t_1)} &= \mathbf{p}_1\mathbf{q}_1^3+((t_1+t_2+1)\mathbf{p}_1+(\lambda_0+\lambda_1-1))\mathbf{q}_1^2\\ &\quad-((t_1+t_2+t_1t_2)\mathbf{p}_1-(2\lambda_0+2\lambda_1-1)(t_1+t_2)-2t_2+2(\lambda_0-1))\frac{\mathbf{q}_1}{2}\\ &\quad+(-(2\lambda_0-1)t_1t_2\mathbf{p}_1+2(\lambda_0+\lambda_1-1)t_2+2\lambda_0-1)t_1+2(\lambda_0-3)t_2\,.\end{aligned}$$

Proposition 3.3.3. *Let q_1, q_2 be the algebraic functions defined in Subsection 3.3.2; then there exist two algebraic functions $p_1(t_1,t_2)$ and $p_2(t_1,t_2)$ such that (q_1,q_2,p_1,p_2) is a solution of (E3.5).*

Proof. Since we know no rational parametrisation of (q_1,q_2) we consider the "symmetrised" system:

$$\begin{cases}\partial_{t_k}S_{\mathbf{q}} &= (\partial_{\mathbf{p}_1}+\partial_{\mathbf{p}_2})H_k & k=1,2\\ \partial_{t_k}P_{\mathbf{q}} &= (\mathbf{q}_2\partial_{\mathbf{p}_1}+\mathbf{q}_1\partial_{\mathbf{p}_2})H_k & k=1,2\\ \partial_{t_k}S_{\mathbf{p}} &= -(\partial_{\mathbf{q}_1}+\partial_{\mathbf{q}_2})H_k & k=1,2\\ \partial_{t_k}\gamma &= \dfrac{-1}{(\mathbf{q}_1-\mathbf{q}_2)^2}((\mathbf{q}_1-\mathbf{q}_2)(\partial_{\mathbf{q}_1}+\partial_{\mathbf{q}_2})+(\mathbf{p}_1-\mathbf{p}_2)(\partial_{\mathbf{p}_1}+\partial_{\mathbf{p}_2}))H_k & k=1,2\end{cases},$$

where $S_{\mathbf{p}} := \mathbf{p}_1+\mathbf{p}_2$ and $\gamma = \dfrac{\mathbf{p}_1-\mathbf{p}_2}{\mathbf{q}_1-\mathbf{q}_2}$. To obtain this we first had to consider the variable $\delta := \mathbf{q}_1-\mathbf{q}_2$ and then eliminate it using the fact that all expressions obtained had even degree in δ and that $\delta^2 = S_{\mathbf{q}}^2-4P_{\mathbf{q}}$.

Assume that (p_1,p_2) are two algebraic functions such that (q_1,q_2,p_1,p_2) is a solution of (E3.5). Using the first two equations with $k=1$ one then gets $S_{\mathbf{p}}$ and γ as functions of $\partial_{t_1}S_q$ and $\partial_{t_1}P_q$ which in turn (see Lemma 3.3.2) are rational functions of (α,t), namely:

$$\begin{aligned}\gamma &= -\frac{(\lambda_0+\lambda_1-1)(\alpha+1)(\alpha-z^2)^2(\alpha-1)}{2\alpha(\alpha-z)(\alpha+z)(z+1)(z-1)}\,,\\ S_p &= \frac{(\alpha-z^2)}{2\alpha(\alpha-z)(\alpha+z)(z+1)(z-1)}\hat{S}_p\,,\end{aligned}$$

with

$$\begin{aligned}\hat{S}_p =&(\lambda_0+2\lambda_1-1)\alpha^3\\ &+((2\lambda_0+\lambda_-2)z^2-(3\lambda_0+\lambda_1-3))\alpha^2\\ &+((\lambda_0-3\lambda_1+1)\alpha+(\lambda_0-1))z^2\,.\end{aligned}$$

This completes the rational parametrisation of all relevant variables and allows us check that (S_q,P_q,S_p,γ) indeed satisfies the above system. ☐

We can describe more precisely the rational surface parametrising q_1 and q_2 as follows. Using the equations linking (S_t,P_t) to (P_q,S_q) and Gröbner bases one show that S_q is root of a degree four polynomial with coefficients depending on S_t, P_t (and thus on t_1,t_2)

and that P_q can be computed as a polynomial in S_t, P_t and S_q. Therefore, there exists a polynomial $P \in \mathbb{C}[X, T_1, T_2]$ of degree four in its first variable such that $P(S_q, t_1, t_2) = 0$ and so if one sets

$$\Sigma := \{x, t_1, t_2 \in \mathbb{P}^1 \mid P(x, t_1, t_2) = 0\}$$

then the projection $p : \Sigma \to \mathbb{P}^1_{t_1} \times \mathbb{P}^1_{t_2}$ is a fourfold ramified cover, whose holonomy we can fully describe.

Proposition 3.3.4. *The holonomy representation into $\mathfrak{S}_4$ of the cover p is trivial at $t_1 = t_2$ and is a double transposition at $t_i = 0, 1, \infty$ $(i = 1, 2)$.*

Proof. Since (S_q, P_q) is solution of a Garnier system, we know that this cover can only ramify over $t_i = 0, 1, \infty$ $(i = 1, 2)$ or $t_1 = t_2$. To better understand the way it does, let us look into its holonomy representation, which is a mapping from the fundamental group G of the complement of the ramification locus in $\mathbb{P}^1 \times \mathbb{P}^1$ into the symmetric group $\mathfrak{S}_4$. By explicitly factorising the polynomial P over all components of the possible ramification locus one gets that:

- over $t_i = 0$ $(i = 1, 2)$ the polynomial has two double roots;
- over $t_i = 1$ $(i = 1, 2)$, the situation is the same
- over $t_i = \infty$ $(i = 1, 2)$, there is only one order four root;
- over $t_1 = t_2$ the polynomial has four simple roots (the cover doesn't actually ramify there).

If one looks (for example) at the restricted polynomial $P(Sq, t_1, 7)$ one can see that its discriminant has a double root at $t_1 = 1$ and that the same is true should one exchange the roles of t_1 and t_2; this means that the holonomy around $t_i = 0, 1$ is a double transposition. Moreover, it takes two elementary transforms to turn the ramification at infinity into two double roots with the discriminant in Sq having a double root there. The holonomy being invariant under birational morphisms, it is also a double transposition. □

Corollary 3.3.4.1. *The complex surface Σ is rational.*

Proof. By setting t_1 or t_2 to any value distinct from $0, 1, \infty$, one gets a fourfold cover from some curve C onto $\mathbb{P}^1$ ramifying over $0, 1$ and ∞. The Riemann–Hurwitz formula yields that the curve C is of genus zero, meaning that it is necessarily a rational curve. This proves that the surface Σ is a fibration over $\mathbb{P}^1$ with general fibre isomorphic to $\mathbb{P}^1$ and so is in fact rational (see for example [39]). □

3.4 Lotka–Volterra foliations

In order to prove Theorem E, let us first define the following notion (see [53]).

Definition 3.4.1 (Transversally projective foliation)**.** *Let M be a smooth projective complex manifold; a codimension one foliation $\mathcal{F}$ on M (defined by a Frobenius–integrable nonzero rational one–form $\omega_{\mathcal{F}}$) is said to be* transversally projective *if there exist two rational one–forms α, β over M such that*

$$\mathrm{d} + \begin{pmatrix} \alpha & \beta \\ \omega_{\mathcal{F}} & -\alpha \end{pmatrix}$$

defines a flat $\mathfrak{sl}_2(\mathbb{C})$–connection over the rank two trivial bundle $\mathbb{C}^2 \times M$.

If one looks at the restriction ω of the Riccati one–form $\mathfrak{R}(\nabla)$ to $(w = \infty)$ one obtains a codimension one transversally projective foliation $\mathcal{F}$ over the projective plane $\mathbb{P}^2$; indeed, if

$$\mathfrak{R}(\nabla) = \mathrm{d}w + \omega w^2 + 2\alpha w + \beta$$

then

$$\mathrm{d} + \begin{pmatrix} \alpha & \beta \\ \omega & -\alpha \end{pmatrix}$$

is gauge–equivalent to ∇ and as such is a flat $\mathfrak{sl}_2(\mathbb{C})$–connection over $\mathbb{C}^2 \times \mathbb{P}^2$. The one–form ω can be written in the affine chart $\mathbb{C}^2_{x,y} \subset \mathbb{P}^2$ described in Subsection 3.2.1 as:

$$\omega = ((2\lambda_0 + \lambda_1)x + \lambda_1(y-1))y\mathrm{d}x - ((\lambda_0 + 2\lambda_1)y + \lambda_0(x-1))x\mathrm{d}y$$

This foliation's invariant locus contains the singular locus of ∇, namely the quintic $\mathcal{Q}$ and has seven order one singularities, namely (in homogeneous coordinates $[x : y : t]$ chosen so that our usual affine chart corresponds to $t = 1$) $[0:0:1]$, $[0:1:1]$, $[1:0:1]$, $[\lambda_1^2 : \lambda_0^2 : (\lambda_0+\lambda_1)^2]$, $[1:1:0]$, [and $[1:0:0]$. Also note that this foliation only depends on the quotient $\lambda := \dfrac{\lambda_0}{\lambda_1}$; indeed it is equivalent to:

$$((2\lambda+1)x + y - 1)y\mathrm{d}x - ((\lambda+2)y + \lambda(x-1))x\mathrm{d}y = 0 \ .$$

Also note that every singular point of the above foliation lies on the quintic $\mathcal{Q}$.

Now define, given three complex parameters (A, B, C), the Lotka-Volterra vector field over $\mathbb{C}^3$ (with coordinates x, y, t) as $\mathrm{LV}(A, B, C) := V_x\partial_x + V_y\partial_y + V_t\partial_t$, where:

$$V_x := x(Cy + t), \qquad V_y := y(At + x) \qquad \text{and} \qquad Vt := t(Bx + y) \ .$$

This system traditionally comes from the study of a "food chain" system with 3 species preying on each other in a cycle. One can then [48, 49] consider the foliation defined by both $\mathrm{LV}(A, B, C)$ and the radial vector field $R := x\partial_x + y\partial_y + t\partial_t$: it is the codimension

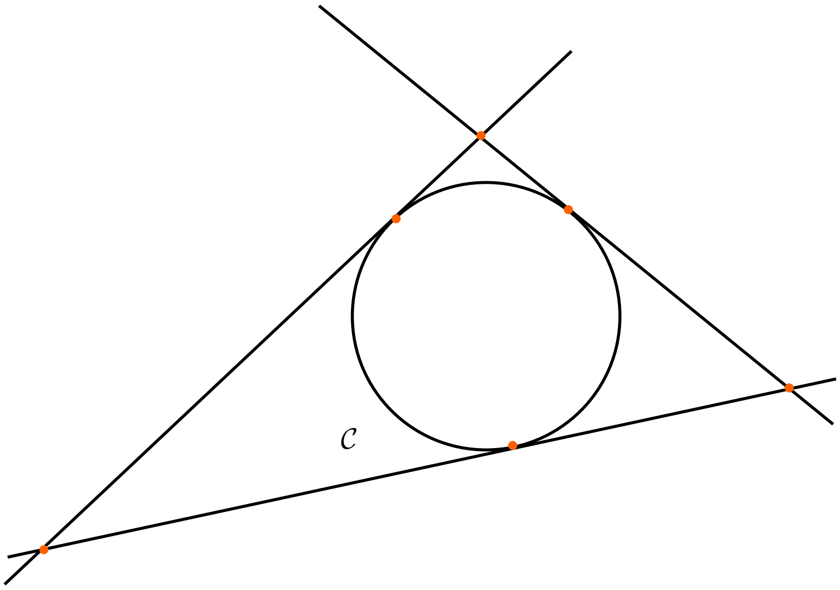

Figure 3.3: Singular locus for the foliation $\mathcal{F}$.

one foliation over $\mathbb{C}^3$ associated with the one–form

$$\omega_0 := (yV_t - tV_y)\mathrm{d}x + (tV_x - xV_t)\mathrm{d}y + (xV_y - yV_x)\mathrm{d}t \ .$$

3.4.1 Proof of Theorem E

To prove Theorem E, one needs only show that the foliations defined by the one–forms ω and $\omega_0' := \omega_{0|_{t=1}}$ are the same in some affine chart. Each of the aforementioned one–forms has four singular points, namely

$$(0,0)\ ,\left(\frac{1}{B},0\right),\ (0,A)\ \text{ and }\left(\frac{A(C-1)+1}{C(B-1)+1},\frac{B(A-1)+1}{C(B-1)+1}\right)\quad \text{for } \omega_0'$$

and

$$(0,0)\ ,(1,0)\ ,\ (0,1)\ \text{ and }\left(\frac{\lambda_1^2}{(\lambda_0+\lambda_1)^2},\frac{\lambda_0^2}{(\lambda_0+\lambda_1)^2}\right)\quad \text{for } \omega\ .$$

We then submit ω_0' to an affine change of coordinates to send its first three singular points onto $(0,0)$, $(1,0)$ and $(0,1)$. A necessary condition for the two forms to define the same foliation is then that their fourth singularities be equal; after computation we find that one must have:

$$\frac{B(A(C-1)+1)}{C(B-1)+1} = \frac{\lambda_1^2}{(\lambda_0+\lambda_1)^2} \tag{E3.6}$$

and

$$\frac{B(A-1)+1}{A(C(B-1)+1)} = \frac{\lambda_0^2}{(\lambda_0+\lambda_1)^2}\ . \tag{E3.7}$$

Solving the two above equations, one obtains a rational parametrisation of A and C by B, namely:

$$A = \frac{(B-1)\lambda_1}{(2\lambda_0+\lambda_1)B} \quad \text{and} \quad C = -\frac{2B(\lambda_0+\lambda_1)^2+\lambda_0\lambda_1}{\lambda_0\lambda_1(B-1)}\ .$$

A necessary and sufficient condition for the two associated foliation to coincide is that $\omega \wedge \omega_0' = 0$; using this and the above parametrisation one gets that B must be equal to $-\dfrac{\lambda_0}{\lambda_0+\lambda_1}$ and thus obtains the first par of Theorem E.

Conversely, direct computation shows that any degree two foliation over $\mathbb{P}^2$ whose invariant locus contains the quintic $\mathcal{Q}$ can be written in the affine chart (s,p) as

$$((\gamma_1+2\gamma_2)x+\gamma_1(y-1))y\mathrm{d}x - ((2\gamma_1+\gamma_2)y+\gamma_2(x-1))x\mathrm{d}y$$

with $\gamma_1, \gamma_2 \in \mathbb{C}$. In particular, such a foliation automatically comes from the monodromy representation of one of our connections $\nabla_{\lambda_0,\lambda_1}$, with $\lambda_0 = \gamma_2$ and $\lambda_1 = \gamma_1$.

Remark 3.4.2.

1. *The relation $ABC = 1$ obtained in Theorem E can be seen intuitively as coming from the order 3 symmetry of the quintic $\mathcal{Q}$: indeed if one denotes by J the homographic order 3 transform defined on $\mathbb{P}^1$ by*

$$z \mapsto -\frac{1}{1+z}$$

then one has

$$(A,B,C) = \left(\frac{\lambda_1}{\lambda_0}, J\left(\frac{\lambda_1}{\lambda_0}\right), J^2\left(\frac{\lambda_1}{\lambda_0}\right)\right)\ .$$

2. *The two variables Lotka–Volterra system is usually defined as being following "prey–predator" differential system:*

$$\begin{cases} x' = x(\alpha+\beta y) \\ y' = y(\gamma+\delta x) \end{cases},$$

to model an ecosystem where x preys on y. However, the plane foliation associated with this system cannot be conjugate to the one associated with ω as it has two double singular points whereas ω has seven simple singularities. Thus this gives some form of justification to the fact that we chose to consider a three variables system in this paragraph (as opposed to the more "natural" two variables one).

3.4.2 Invariant curves

The invariant locus of the family of foliations presented here does not have normal crossings, hence the Cerveau–Lins Neto bound on the degree ($\deg(\mathcal{F})+2$, see [13]) does not apply here. Furthermore one may note that (for generic parameters λ_0, λ_1) the foliation $\mathcal{F}$ has simple singularities at the tangency locus of the conic $\mathcal{C}$ and the three invariant lines. Moreover, we have the following result.

Proposition 3.4.3. *The foliation $\mathcal{F}$ admits, for $\lambda_0, \lambda_1 \in \mathbb{Q}$, invariant algebraic curves of arbitrarily high (depending on λ_0/λ_1) degree.*

Proof. The section $(w = \infty) \subset \mathbb{P}^1 \times \mathbb{P}^2$ that we used to define our foliations lifts through $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{2:1} \mathbb{P}^2$ (see Subsection 3.2.1) to the section $(z = 1)$ of the trivial bundle $X \times \mathbb{P}^1$ (see Subsection 3.2.2) and so the foliation itself lifts (in our usual local chart) to:

$$(\mathcal{F}') \qquad \lambda_0 \left(\frac{\mathrm{d}u_0}{u_0} - \frac{\mathrm{d}u_1}{u_1} \right) + \lambda_1 \left(\frac{\mathrm{d}u_0}{u_0 - 1} - \frac{\mathrm{d}u_1}{u_1 - 1} \right) = 0 \ .$$

If one looks at rational values of λ_0 and λ_1, one gets a foliation $\mathcal{F}$ with finite holonomy which as a consequence admits a rational first integral. Moreover, in that particular case every leaf is an algebraic invariance curve and it is possible to find these with arbitrarily high degree (for varying λ_0, λ_1). For example, if $\lambda_0 = n \geq 1$ is a positive integer and if we set $\lambda_1 = 1$ then a simple computation shows that the curve

$$(C_n) \qquad u_0^n(u_0 - 1) - u_1^n(u_1 - 1) = 0$$

on X is invariant under $\mathcal{F}'$. An induction then shows that this curve is the pullback by π of a degree $2 + n$ curve on $\mathbb{P}^2$ and so we get an invariant curve of such degree for the foliation $\mathcal{F}$ corresponding with the parameters $(n, 1)$. $\square$

Remark 3.4.4. *Note however that this is a slightly weaker example than the ones given in* [41] *as the local type of our singularities depends on the parameter λ_0/λ_1.*

3.5 Proof of Theorem D

In this paragraph, we prove that our family of monodromy representations cannot be generically obtained through a pullback method [24, 25] by showing that it does not factor through an orbicurve [15].

3.5.1 First case: λ_0 and λ_1 are not linearly dependant over $\mathbb{Z}$

Suppose that we have some complex projective curve C, a divisor $\delta = t_1 + \ldots + t_k$ in C, an algebraic mapping $f : \mathbb{P}^2 - f^{-1}(\delta) \to C - \delta$ and a representation $\tilde{\rho}$ of the fundamental group of $C - \delta$ into $PSL_2(\mathbb{C})$ such that the diagram

$$\begin{tikzcd} \pi_1(C - \delta, x_0) \arrow[dr, "\tilde{\rho}"] & \pi_1(\mathbb{P}^2 - f^{-1}(\delta)) \arrow[l, "f_*"'] \arrow[d, "\mathrm{P} \circ \rho \circ m"] \\ & PSL_2(\mathbb{C}) \end{tikzcd}$$

commutes. Since the ramified cover $\pi : X \xrightarrow{2:1} \mathbb{P}^2$ is unramified between $X - D$ and $\mathbb{P}^2 - \mathcal{Q}$, where D is the divisor in X made of the six lines $u_0, u_1 = 0, 1, \infty$ and the diagonal $\Delta = (u_0 = u_1)$, then the fundamental group $\pi_1(X - D)$ is realised as a subgroup of Γ_2. This means that if one sets $\phi := f \circ \pi$ one has such a diagram:

$$\begin{tikzcd} \pi_1(C-\delta, x_0) \arrow[dr, "\tilde{\rho'}"] & \pi_1(X - \phi^{-1}(\delta)) \arrow[l, "\phi_*"'] \arrow[d, "\rho'"] \\ & PSL_2(\mathbb{C}) \end{tikzcd} .$$

Now let L be a generic horizontal line in X (i.e of the form $(u_1 = c)$, with $c \neq 0, 1, \infty$); since f is algebraic the restricted map $\phi_{|_L}$ extends as a ramified cover $\phi_L : L \to C$ with topological degree equal to some $d \geq 1$. The line L is isomorphic to $\mathbb{P}^1$, so the Riemann–Hurwitz formula forces the genus of the curve C to be equal to zero; as such we can assume without loss of generality that ϕ_L is a d–fold cover of the projective line over itself. Moreover, one has that $\phi_L^* \delta$ must contain $\{0, 1, c, \infty\}$.

The representation $\tilde{\rho'}$ must induce infinite order monodromy about at least one loop in $C - \delta$, say γ_0, or else all elements in the image of ρ would be of finite order. This means that $M := \tilde{\rho'}(\gamma_0)$ is a infinite-order element in $PSL_2(\mathbb{C})$.

Let us assume that there are at least two distinct elements γ and γ' in the fibre of $(\phi_L)_*$ above γ_0; then both $\rho'(\gamma)$ and $\rho'(\gamma')$ must be powers of M. This gives us a relation between words in the matrices

$$\begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix}, \qquad \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} a_0 a_1 & 0 \\ 0 & (a_0 a_1)^{-1} \end{pmatrix} \quad ,$$

where $a_j = e^{-i\pi\lambda_j}$. Since generically λ_0 and λ_1 are not linearly dependant, this is impossible; hence we have that the fibre $(\phi_L)_*^{-1}(\gamma_0)$ may only contain one element. This implies that ϕ_L ramifies totally over (at least) three points in C and so the Riemann–Hurwitz formula yields that ϕ_L must be one–to–one.

Let $u \in \mathbb{P}^1$ and set $h_u \in PSL_2(\mathbb{C})$ to be the Möbius transform sending the ramification locus of $\phi_{(u_1 = u)}$ onto $0, 1, \infty$; up to composing it with $(u_0, u_1) \mapsto (h_{u_1}(u_0), u_1)$ we can assume that ϕ is exactly the first projection $\mathrm{pr}_1 : X \to \mathbb{P}^1$. However if one looks at the restriction of ϕ to some vertical line then one should again generically obtain infinite local monodromy at three points, which is impossible with pr_1, thus concluding the proof.

3.5.2 Second case: there exists (p, q) in $\mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $p\lambda_0 + q\lambda_1 = 0$

We can assume that at least one of $\frac{\lambda_0}{\lambda_1}$ or $\frac{\lambda_1}{\lambda_0}$ is a rational number, therefore the transversally projective foliation $\mathcal{F}$ introduced in Section 3.4 has finite monodromy and so admits some rational first integral $g : \mathbb{P}^2 \to \mathbb{P}^1$. Using Subsection 4.4 in [43], one deduces that the transversally projective structure (β, α, ω) associated with $\mathcal{F}$ is equivalent to one of the form $(\tilde{\beta}, 0, \mathrm{d}g)$ with the following relations (see [43], Subsection 4.1):

$$\tilde{\beta} \wedge \mathrm{d}g = 0 \quad \text{and} \quad \mathrm{d}\tilde{\beta} = 0 \; .$$

The first relation implies that $\tilde{\beta}$ must be of the form $\tilde{\beta} = f\mathrm{d}g$ for some rational $f : \mathbb{P}^2 \to \mathbb{P}^1$; using the second relation one then gets that

$$\mathrm{d}f \wedge \mathrm{d}g = 0 \; . \tag{E3.8}$$

Using standard results from birational geometry (see for example Theorem II.7 in [4]) one obtains that there exists a complex surface M and a finite sequence $\mathfrak{b} : M \to \mathbb{P}^2$ of blow–ups such that $\mathfrak{g} := g \circ \mathfrak{b}$ is a holomorphic function on M. Moreover, if we set $\mathfrak{f} := f \circ \mathfrak{b}$ then we must have

$$\mathrm{d}\mathfrak{f} \wedge \mathrm{d}\mathfrak{g} = 0 \; . \tag{E3.9}$$

It then follows from Stein's factorisation theorem that there exists a complex curve C, a ramified cover $r : C \to \mathbb{P}^1$ and a fibration $\phi : M \to C$ with connected fibres such that the following diagram

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\mathfrak{g}"'] & C \arrow[dl, "r"] \\ \mathbb{P}^1 & \end{tikzcd}$$

commutes. This means that locally on any sufficiently small analytic open set U the cover r gives an orbifold coordinate x on the curve C and there exists a biholomorphism h between $U \times F$ and $\phi^{-1}(U)$, where F is a connected complex curve, such that for all $(x, y) \in U \times F$, $g \circ h(x, y) = x$. Therefore relation (E3.9) yields:

$$\mathrm{d}(\mathfrak{f} \circ h) \wedge \mathrm{d}x = 0 \; .$$

Thus $\mathfrak{f}$ depends locally only on $\mathfrak{g}$ and since the fibres of ϕ are connected one can conclude using analytic continuation that $\mathfrak{f}$ is globally a function of $\mathfrak{g}$. In the end, this implies that the transversally projective structure associated with $\mathcal{F}$ is equivalent to $(f(g)\mathrm{d}g, 0, \mathrm{d}g)$ and so factors through through the algebraic map associated with f on $\mathbb{P}^2 - I$, where I is the indeterminacy locus of f.

Chapter 4

Second family of solutions

The object of this paragraph is to give an explicit construction of a family of logarithmic flat connections on $\mathbb{P}^2_{\mathbb{C}}$ whose monodromy realises the second representation in Theorem A (see Chapter 1) and to describe the associated isomonodromic deformation of the five punctured sphere. In order to do so, we will follow broadly the method outlined in Chapter 3; without going into as much detail since both constructions are quite similar.

4.1 Rank two connected bundle over the projective plane

4.1.1 Set–up

Consider the projective plane quintic curve $\mathcal{Q}' \subset \mathbb{P}^2_{\mathbb{C}}$ defined (in homogeneous coordinates $[x : y : z]$) by the equation:

$$y(y-z)z(x^2-yz) = 0 \ .$$

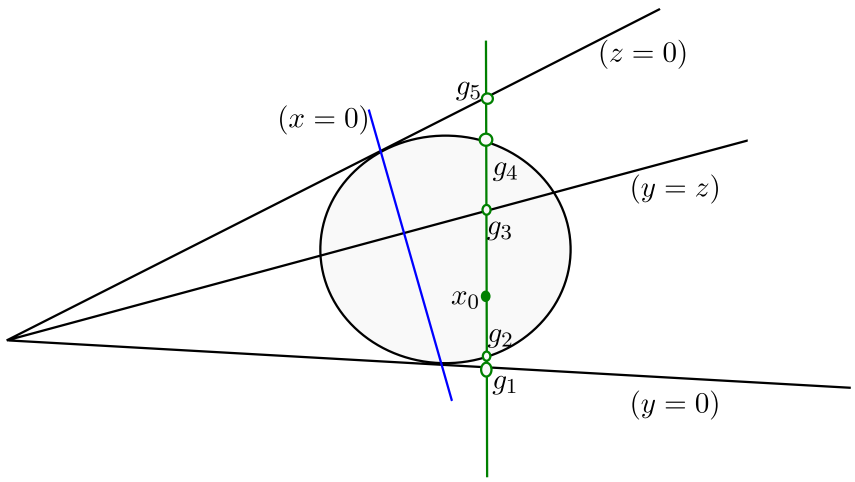

Figure 4.1: Fundamental group of $\mathbb{P}^2 - \mathcal{Q}'$ and restriction to a generic line.

We know from Degtyarev's list (see Chapter 1) that the fondamental group of the complement of the quintic $\mathcal{Q}'$ is isomorphic to:

$$\Gamma_2' := \langle a, b, c \,|\, [a,b] = [a, c^{-1}bc] = 1,\ (bc)^2 = (cb)^2 \rangle \ .$$

More precisely, we get from the Zariski–Van Kampen method that we can take c (resp. a, b) to be a loop realising the local monodromy around the conic $\mathcal{C}'$ defined by the equation $x^2 - yz = 0$ (resp. the lines $(y = z)$, $(y = 0)$). The Lefschetz hyperplane theorem (see [47], Theorem 7.4) tells us that the natural morphism $\tau : \mathbf{F}_4 \to \Gamma_2'$ stemming from restriction to a generic line is onto and if we chose a line passing through the base point used to define a, b and c then τ is given (up to changing the generators g_i) by (see Figure 4.1):

$$\begin{aligned} g_1 &\mapsto c \\ g_2 &\mapsto b \\ g_3 &\mapsto a \\ g_4 &\mapsto b \\ g_5 &\mapsto (cbab)^{-1} = (cb^2a)^{-1} \ . \end{aligned}$$

As per Theorem A, we wish to construct a family of logarithmic flat connections over $\mathbb{P}^2$ with polar locus equal to $\mathcal{Q}'$ and monodromy of the form:

$$\rho_{u,v} : a \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \text{ for } u, v \in \mathbb{C}^* \ .$$

More precisely, we prove the following result.

Theorem F. *There exists an* explicit *two–parameter family $\nabla_{\lambda_0,\lambda_1}$ of logarithmic flat connections over the trivial rank two vector bundle $\mathbb{C}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ with the following properties:*

(i) the polar locus of $\nabla_{\lambda_0,\lambda_1}$ is equal to the quintic $\mathcal{Q}' \in \mathbb{P}^2$ defined by the equation

$$y(y-t)t(x^2 - yt) = 0 \ \ ;$$

(ii) the monodromy of $\nabla_{\lambda_0,\lambda_1}$ is conjugated to $\rho_{u,v}$ with $u = e^{i\pi\lambda_0}$ and $v = e^{i\pi\lambda_1}$. It is a virtually abelian dihedral representation of the fundamental group $\Gamma_2 := \pi_1(\mathbb{P}^2 - \mathcal{Q})$ into $SL_2(\mathbb{C})$ whose image is not Zariski–dense.

The connection $\nabla_{\lambda_0,\lambda_1}$ is given in the affine chart $\mathbb{C}^2_{x,y} \subset \mathbb{P}^2$ by:

$$\nabla_{\lambda_0,\lambda_1} = \mathrm{d} - \frac{1}{y(y-1)(x^2-y)} \Omega_{\lambda_0,\lambda_1} \ ,$$

where

$$\Omega_{\lambda_0,\lambda_1} := \begin{pmatrix} -\frac{(y-1)(x^2-y)}{4y}\mathrm{d}y & -\frac{2\lambda_0 y(y-1)\mathrm{d}x + (\lambda_0 x(1-y) + \lambda_1(x^2-y))\mathrm{d}y}{2} y \\ -\frac{2\lambda_0 y(y-1)\mathrm{d}x + (\lambda_0 x(1-y) + \lambda_1(x^2-y))\mathrm{d}y}{2} & \frac{(y-1)(x^2-y)}{4y}\mathrm{d}y \end{pmatrix} .$$

Moreover, the monodromy representation of such a connection factors through an orbicurve if and only if there exists $(p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $p\lambda_0 + q\lambda_1 = 0$.

Remark 4.1.1. *Note that, similar to Chapter 3, the existence, and uniqueness up to gauge transformation, of such a family of connections follows from the classical Riemann–Hilbert correspondence.*

4.1.2 A suitable double cover

As in Chapter 3, the key point of our construction will be to find a properly ramified double cover so that we are able to obtain a connection with dihedral monodromy. Here, since the local monodromy with projective order two arises along the line $\{y = 0\}$, we will need to have the aforementioned cover ramify there. Consider the following $2:1$ birational map:

$$\begin{aligned} \pi : \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^2 \\ ([u_0 : u_1], [v_0, v_1]) &\mapsto [u_0 v_1^2 : u_1 v_0^2 : u_1 v_1^2] \,. \end{aligned}$$

The map π has indeterminacy locus equal to the point $\{([1:0],[1:0])\}$ and ramifies over $\{y = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$. Moreover, one also has (see Fig. 4.2):

$$\begin{aligned} \pi^* \mathcal{C}' &= \{([u_0 : u_1],[v_0 : v_1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \,|\, (u_0 v_1 - u_1 v_0)(u_0 v_1 + u_1 v_0) = 0\} \\ &= \{u = v\} \cup \{u = -v\} \subset \mathbb{P}^1_u \times \mathbb{P}^1_v \,, \end{aligned}$$

$$\begin{aligned} \pi^*(\{y = z\} \cap \{z \neq 0\}) &= \{([u_0 : u_1],[v_0 : v_1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \,|\, v_0^2 = v_1^2\} \\ &= \{v = 1\} \cup \{v = -1\} \subset \mathbb{P}^1_u \times \mathbb{P}^1_v \,, \end{aligned}$$

$$\begin{aligned} \pi^*(\{y = 0\} \cap \{z \neq 0\}) &= \{([u_0 : u_1],[v_0 : v_1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \,|\, v_0 = 0\} \\ &= \{v = 0\} \subset \mathbb{P}^1_u \times \mathbb{P}^1_v \end{aligned}$$

and

$$\begin{aligned} \pi^*\{z = 0\} &= \{([u_0 : u_1],[v_0 : v_1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \,|\, u_1 v_1^2 = 0\} \\ &= \{u = \infty\} \cup \{v = \infty\} \subset \mathbb{P}^1_u \times \mathbb{P}^1_v \,. \end{aligned}$$

This means that the quintic $\mathcal{Q}'$ is pulled back by π onto seven lines in $\mathbb{P}^1 \times \mathbb{P}^1$. The cover π corresponds to the quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ under the involution $(u,v) \mapsto (u,-v)$. Now let us consider the elementary transformation of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by:

$$\begin{aligned} \mathfrak{b} : \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^1 \times \mathbb{P}^1 \\ ([u_0 : u_1],[v_0 : v_1]) &\mapsto ([u_0 v_0 : u_1 v_1],[v_0 : v_1]) \,; \end{aligned}$$

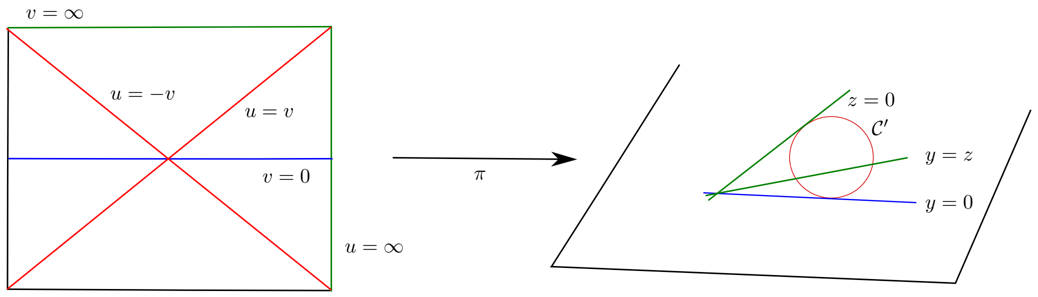

Figure 4.2: Ramification locus of π.

then one can easily check that:

$$\begin{aligned}\mathfrak{b}^*(\{u=\pm v\}\cap\{v\neq 0,\infty\}) &= \{([u_0:u_1],[v_0:v_1])\in\mathbb{P}^1\times\mathbb{P}^1 \,|\, (u_0v_0-\pm v_0)(u_1v_1-\pm v1)=0\}\\ &= \{u=\pm 1\}\subset \mathbb{P}^1_u\times\mathbb{P}^1_v\end{aligned}$$

and

$$\begin{aligned}\mathfrak{b}^*\{u=\infty\} &= \{([u_0:u_1],[v_0,v_1])\in\mathbb{P}^1\times\mathbb{P}^1 \,|\, u_1v_1=0\}\\ &= \{u=\infty\}\cup\{v=\infty\}\cup\{v=\infty\}\subset\mathbb{P}^1_u\times\mathbb{P}^1_v\,.\end{aligned}$$

The heuristic behind this procedure can be summed up as follows: imagine that we have a logarithmic flat connection ∇ over $\mathbb{P}^2$ satisfying the hypotheses of Theorem F; then $\pi^*\nabla$ will be a logarithmic flat connection over $\mathbb{P}^1\times\mathbb{P}^1$ whose monodromy around $\{v=0\}$ must be (projectively) trivial since π ramifies over $\{y=0\}$. As such, its monodromy group must be abelian and the pullback connection $(\pi\circ\mathfrak{b})^*\nabla$ also has abelian monodromy and factors through the fundamental group of the complement of

$$\{u=\infty\}\cup\{v=\infty\}\cup\{u=1\}\cup\{u=-1\}\cup\{v=1\}\cup\{v=-1\}\text{ in }\mathbb{P}^1_u\times\mathbb{P}^1_v$$

which is a product of free groups. This will allow us to set up a construction quite similar to that of Chapter 3.

4.1.3 Constructing the connection

Drawing inspiration from the above heuristic and Chapter 3, we consider the trivial rank two vector bundle E_0 on $X:=\mathbb{P}^1\times\mathbb{P}^1$ endowed with the logarithmic flat connection:

$$\nabla_0 := \mathrm{d}+\frac{1}{2}\begin{pmatrix}\omega_0 & 0\\ 0 & -\omega_0\end{pmatrix}\,,$$

where u, v are projective coordinates on the base X and

$$\omega_0 := \lambda_0\left(\frac{\mathrm{d}u}{u-1}-\frac{\mathrm{d}u}{u+1}\right)+\lambda_1\left(\frac{\mathrm{d}v}{v-1}-\frac{\mathrm{d}v}{v+1}\right),$$

with $(\lambda_0,\lambda_1)\in\mathbb{C}^2\setminus\{(0,0)\}$. This connection generically has singular locus equal to four lines in X and its local monodromy is given by the following matrices:

- around $\{u=\pm 1\}$:
$$\begin{pmatrix} e^{-i\pi\lambda_0} & 0 \\ 0 & e^{i\pi\lambda_0}\end{pmatrix}^{\pm 1};$$

- around $\{v=\pm 1\}$, $j=0,1$:
$$\begin{pmatrix} e^{i\pi\lambda_j} & 0 \\ 0 & e^{i\pi\lambda_j}\end{pmatrix}^{\pm 1}.$$

One easily checks that the one–form ω_0 is the pullback under the elementary transform $\mathfrak{b}$ of:

$$\omega_1 := \lambda_0\left(\frac{\mathrm{d}u}{u-v}-\frac{\mathrm{d}u}{u+v}+\frac{u}{v}\left(\frac{\mathrm{d}v}{u+v}-\frac{\mathrm{d}v}{u-v}\right)\right)+\lambda_1\left(\frac{\mathrm{d}v}{v-1}-\frac{\mathrm{d}v}{v+1}\right).$$

Note that this one–form gets non–trivial local monodromy around the lines $\{v=0\}$ and $\{u=\infty\}$; it naturally gives rise to a Riccati foliation, defined by the following one–form over $\mathbb{P}(E_0)=X\times\mathbb{P}^1$:

$$\mathfrak{R}_1 := \mathrm{d}w-\omega_1 w$$

where w is a projective coordinate on the fibres.

Now define the following involution on the projective bundle $\mathbb{P}(E_0)$:

$$\begin{aligned}\eta : X\times\mathbb{P}^1 &\to X\times\mathbb{P}^1\\ (u,v,w)&\mapsto(-u,-v,-w)\end{aligned}$$

that leaves invariant the one–form $\mathfrak{R}_1$. This means that if we extend π into the map

$$\begin{aligned}\bar{\pi} : X\times\mathbb{P}^1 &\to \mathbb{P}^2\times\mathbb{P}^1\\ (u,v,[w_0:w_1])&\mapsto(\pi(u,v),[w_0+w_1:w_0-w_1])\end{aligned}$$

then $\mathfrak{R}_1$ is the pullback under $\bar{\pi}$ of the Riccati one–form defined in some affine chart by:

$$\begin{aligned}\mathfrak{R} := \mathrm{d}w&+\left(\frac{\lambda_0}{x^2-y}\mathrm{d}x+\left(\frac{\lambda_0 x}{x^2-y}-\frac{\lambda_1}{2(y-1)}\right)\frac{\mathrm{d}y}{y}\right)w^2\\ &-\frac{\mathrm{d}y}{2y}w-\frac{\lambda_0 y}{x^2-y}\mathrm{d}x+\frac{1}{2}\left(\frac{\lambda_0 x}{y(x^2-y)}-\frac{\lambda_1}{y-1}\right)\mathrm{d}y\,.\end{aligned}$$

The above is a logarithmic one–form over $\mathbb{P}^2\times\mathbb{P}^1$ whose singular locus is exactly the

Divisor	Residue	Eigenvalues
$y=0$	$\begin{pmatrix} -\frac{1}{4} & 0 \\ \frac{\lambda_0+\lambda_1 x}{2x} & \frac{1}{4} \end{pmatrix}$	$\pm\frac{1}{4}$
$y=1$	$\begin{pmatrix} 0 & -\frac{\lambda_1}{2} \\ -\frac{\lambda_1}{2} & 0 \end{pmatrix}$	$\pm\frac{\lambda_1}{2}$
$\mathcal{C}$	$\begin{pmatrix} 0 & \frac{\lambda_0 x}{2} \\ \frac{\lambda_0}{2x} & 0 \end{pmatrix}$	$\pm\frac{\lambda_0}{2}$
L_∞	$\begin{pmatrix} -\frac{1}{4} & 0 \\ \frac{\lambda_1}{2} & \frac{1}{4} \end{pmatrix}$	$\pm\frac{1}{4}$

Table 4.1: Residues for ∇.

quintic $\mathcal{Q}'$. Moreover, this lifts (as explained in Chapter 3) to a logarithmic flat connection $\nabla = \nabla_{\lambda_0,\lambda_1}$ over the trivial bundle $\mathbb{C}^2 \times \mathbb{P}^2$ over the projective plane $\mathbb{P}^2$, namely

$$\nabla = \mathrm{d} + \begin{pmatrix} -\frac{\mathrm{d}y}{4y} & -\frac{\lambda_0 y}{x^2-y}\mathrm{d}x + \frac{1}{2}\left(\frac{\lambda_0 x}{y(x^2-y)} - \frac{\lambda_1}{y-1}\right)\mathrm{d}y \\ \frac{\lambda_0}{x^2-y}\mathrm{d}x + \left(\frac{\lambda_0 x}{x^2-y} - \frac{\lambda_1}{2(y-1)}\right)\frac{\mathrm{d}y}{y} & \frac{\mathrm{d}y}{4y} \end{pmatrix};$$

see table 4.1 for the exact residues.

The fact that this does not generically factor through an orbicurve can then be proven using exactly the same argument as in Section 3.5.1. Indeed, there generically is no algebraic relations between the coefficients of the matrices in table 4.1.

4.2 Associated isomonodromic deformation of the five punctured sphere

4.2.1 Restriction to generic lines

We now fix $\lambda_0, \lambda_1 \in \mathbb{C}^*$ and consider the connection induced by ∇ on generic lines in $\mathbb{P}^2$, such a line being given in the affine chart $\{z=0\}$ by an equation of the form $y = ax+b$. As in Chapter 3, we obtain an isomonodromic deformation $(\nabla_{a,b})_{a,b}$ over the five punctured sphere whose associated Riccati one–form is given by the following local formula (in the aforementioned affine chart):

$$\begin{aligned}\mathfrak{R}(\nabla_{a,b}) =& \frac{2a^2y^4 + (4ab - 2 - 2a^2)y^3 + (2 - 4ab + 2b^2)y^2 - 2b^2y}{2y(y-1)(a^2y^2 + (2ab-1)y + b^2)}\mathrm{d}w \\ &+ \frac{\alpha_2(y)w^2 + \alpha_1(y)w + \alpha_0}{2y(y-1)(a^2y^2 + (2ab-1)y + b^2)}\mathrm{d}y\end{aligned}$$

where

$$\begin{aligned}\alpha_2(y) &:= (\lambda_0 a + \lambda_1 a^2)y^2 + ((a-b)\lambda_0 + (2ab-1)\lambda_1)y + \lambda_0 b + \lambda_1 b^2 \\ \alpha_1(y) &:= -a^2y^3 + (a^2 - 2ab + 1)y^2 + (2ab - b^2 - 1)y + b^2 \\ \alpha_0(y) &:= -(\lambda_0 a + \lambda_1 a^2)y^3 + ((a+b)\lambda_0 + (1-2ab)\lambda_1)y^2 - (\lambda_0 b + \lambda_1 b^2)y \ .\end{aligned}$$

As such, if one chooses a parameter c such that $c^2 = 1 - 4ab$ then one gets a family of logarithmic flat connections $(\nabla_{a,c})_{a,c}$ over $\mathbb{P}^1 \setminus \{0, 1, t_1, t_2, \infty\}$, where:

$$t_1 = \left(\frac{c-1}{2a}\right)^2 \quad \text{and} \quad t_2 = \left(\frac{c+1}{2a}\right)^2 .$$

An explicit local expression for this isomonodromic deformation can be obtained simply by replacing b by $\dfrac{1-c^2}{4a}$ in the above formula for $\mathfrak{R}(\nabla_{a,b})$; this allows us to explicitly compute the spectral data for $\nabla_{a,c}$, as detailed in Table 4.2.

4.2.2 Associated Garnier solution

Similar to what we did in Section 3.3.2, start by setting

$$\hat{H} := \frac{M_0}{y} + \frac{M_1}{y-1} + \frac{M_2}{y-t_1} + \frac{M_3}{y-t_2} \ ;$$

then since the lower left coefficient of the residue at infinity M_∞ of $\nabla_{a,c}$ is zero, the numerator of $\hat{H}$ must be a degree two polynomial in y, say:

$$\hat{H}_{2,1} = \frac{c(t_1,t_2)(y^2 - S_q(t_1,t_2)y + P_q(t_1,t_2))}{y(y-1)(y-t_1)(y-t_2)} \ , \tag{E4.1}$$

where $S_q := q_1 + q_2$ and $P_q := q_1q_2$, with q_1, q_2 some algebraic functions of (t_1, t_2). In the same manner that we did in Chapter 3, we obtain a rational parametrisation of S_q, P_q, t_1, t_2.

More precisely, the parameters (a, c) give a rational mapping $(\mathbb{P}^1)^2 \to (\mathbb{P}^1)^4$ giving

Pole	Residue	Eigenvalues
$y=0$	$M_0 := \begin{pmatrix} -\frac{1}{4} & 0 \\ -\frac{4a\lambda_0-(c^2-1)\lambda_1}{2(c^2-1)} & \frac{1}{4} \end{pmatrix}$	$\pm\frac{1}{4}$
$y=1$	$M_1 := \begin{pmatrix} 0 & -\frac{\lambda_1}{2} \\ -\frac{\lambda_1}{2} & 0 \end{pmatrix}$	$\pm\frac{\lambda_1}{2}$
$y=t_1$	$M_2 := \begin{pmatrix} 0 & \frac{(c-1)\lambda_0}{4a} \\ \frac{a\lambda_0}{c-1} & 0 \end{pmatrix}$	$\pm\frac{\lambda_0}{2}$
$y=t_2$	$M_3 := \begin{pmatrix} 0 & -\frac{(c+1)\lambda_0}{4a} \\ -\frac{a\lambda_0}{c+1} & 0 \end{pmatrix}$	$\pm\frac{\lambda_0}{2}$
$y=\infty$	$M_\infty := \begin{pmatrix} \frac{1}{4} & \frac{\lambda_0+a\lambda_1}{2a} \\ 0 & -\frac{1}{4} \end{pmatrix}$	$\pm\frac{1}{4}$

Table 4.2: Residues for $\nabla_{a,c}$.

explicit expressions of (t_1, t_2, S_q, P_q), namely:

$$\begin{aligned}
t_1 &= \frac{(c-1)^2}{4} \\
t_2 &= \frac{(c+1)^2}{4}\,, \\
S_q &= \frac{(1-c^2+4a^2)\lambda_0 + 2a(c^2+1)\lambda_1}{4a^2(\lambda_0 + a\lambda_1)}\,, \\
P_q &= -\frac{(c-1)(c+1)(4a\lambda_0 + (1-c^2))}{16a^3(\lambda_0 + a\lambda_1)}\,.
\end{aligned}$$

We can now prove that we have indeed constructed a family of algebraic solutions for a Garnier system. More precisely, consider the following Hamiltonian system:

$$\begin{cases} \partial_{t_k}\mathbf{p}_i &= -\partial_{\mathbf{q}_i} H_k \quad i,k = 1,2 \\ \partial_{t_k}\mathbf{q}_i &= \partial_{\mathbf{p}_i} H_k \quad i,k = 1,2 \end{cases}, \tag{E4.2}$$

where the pair of Hamiltonians (H_1, H_2) is explicitly given in Appendix B. Note that it differs slightly from the one presented in Chapter 3, as the local eigenvalues are not the same.

Proposition 4.2.1. *Let q_1, q_2 be the algebraic functions defined above; then there exist two algebraic functions $p_1(t_1, t_2)$ and $p_2(t_1, t_2)$ such that (q_1, q_2, p_1, p_2) is a solution of (E4.2).*

Proof. We proceed exactly as we did for the first family of solutions: we consider the "symmetrised" system:

$$\begin{cases}
\partial_{t_k} S_{\mathbf{q}} &= (\partial_{\mathbf{p}_1} + \partial_{\mathbf{p}_2})H_k & k=1,2 \\
\partial_{t_k} P_{\mathbf{q}} &= (\mathbf{q}_2\partial_{\mathbf{p}_1} + \mathbf{q}_1\partial_{\mathbf{p}_2})H_k & k=1,2 \\
\partial_{t_k} S_{\mathbf{p}} &= -(\partial_{\mathbf{q}_1} + \partial_{\mathbf{q}_2})H_k & k=1,2 \\
\partial_{t_k}\gamma &= \frac{-1}{(\mathbf{q}_1-\mathbf{q}_2)^2}((\mathbf{q}_1-\mathbf{q}_2)(\partial_{\mathbf{q}_1} + \partial_{\mathbf{q}_2}) + (\mathbf{p}_1-\mathbf{p}_2)(\partial_{\mathbf{p}_1} + \partial_{\mathbf{p}_2}))H_k & k=1,2
\end{cases},$$

where $S_{\mathbf{p}} := \mathbf{p}_1 + \mathbf{p}_2$ and $\gamma = \dfrac{\mathbf{p}_1 - \mathbf{p}_2}{\mathbf{q}_1 - \mathbf{q}_2}$. To obtain this we first had to consider the variable $\delta := \mathbf{q}_1 - \mathbf{q}_2$ and then eliminate it using the fact that all expressions obtained had even degree in δ and that $\delta^2 = S_{\mathbf{q}}^2 - 4P_{\mathbf{q}}$.

Assume that (p_1, p_2) are two algebraic functions such that (q_1, q_2, p_1, p_2) is a solution of (E4.2). Using the first two equations with $k = 1$ one then gets $S_{\mathbf{p}}$ and γ as functions of $\partial_{t_1} S_q$ and $\partial_{t_1} P_q$ which in turn are rational functions of (a, c), namely:

$$\begin{aligned}
\gamma &= \frac{8(1+a)(\lambda_0 + a\lambda_1)a^3}{(c^2-1)(1+c+2a)(1-c+2a)}\,, \\
Sp &= -\frac{2a(1+3a-c^2+4a^2-3ac^2+4a^3)\lambda_0 + (2a+4a^2+2ac^2)\lambda_1}{(c^2-1)(1+c+2a)(1-c+2a)}\,.
\end{aligned}$$

This completes the rational parametrisation of all relevant variables and allows us to check that (S_q, P_q, S_p, γ) indeed satisfies the above system, in the exact same way we did in Section 3.3.3. $\square$

Part II

Katz's middle convolution and derivatives

Chapter 5

Some new orbits

In this chapter we draw inspiration from earlier work by Boalch [7] on Painlevé VI equations to obtain finite orbits under the mapping class group action on some character variety through Katz's middle convolution. The basic idea is the following : take a finite subgroup G in $SL_3(\mathbb{C})$ and consider a representation $\rho : \mathbf{F}_4 \to SL_3(\mathbb{C})$ whose image is contained in G. Since the latter is finite, so must be the orbit of the class $[\rho] \in \mathrm{Char}_3(0,5)$ under the action of the mapping class group $\mathrm{Mod}(0,5)$. By applying Katz's middle convolution as described in Section I2 of the introduction to this thesis, we get a new finite orbit under the mapping class group action on Char_{d_ρ} for some explicit $d_\rho \geq 1$. Since the upshot here is to study rank two isomonodromic deformations of the five punctured sphere, we shall endeavour to construct ρ so that $d_\rho = 2$.

5.1 Main result

5.1.1 Framework

Start by recalling that if $\rho : \mathbf{F}_4 \to SL_3(\mathbb{C})$ is a group representation of the free group on four generators $\mathbf{F}_4 := \langle d_1, \ldots, d_5 \,|\, d_1 \ldots d_5 = 1 \rangle$ then its middle convolution with respect to some $\underline{\theta} \in (\mathbb{C}^*)^5$ will be a representation $\tilde{\rho} : \mathbf{F}_4 \to SL_{d_\rho}(\mathbb{C})$, with

$$d_\rho = \sum_{i=1}^{5} \mathrm{rg}(I_d - g_i) - d \ ,$$

where $g_i = \frac{1}{\theta_i}\rho(d_i)$ for $i \in [5]$. Our intention here is to use Katz's middle convolution to obtain rank two representation from finite subgroups of $SL_3(\mathbb{C})$, meaning that the matrices h_i and the complex numbers θ_i will need to be chosen so that :

$$\sum_{i=1}^{5} \mathrm{rank}(I_d - g_i) = 5 \ .$$

In order to do so, we will proceed as follows:

- choose some finite subgroup Γ of $SL_3(\mathbb{C})$;
- select (see Section 5.2.1) five matrices $h_1, \ldots, h_5 \in \Gamma$ such that each of them has a double eigenvalue θ_i (*i.e* an eigenvalue that is a double root of its characteristic polynomial) and so that $h_1, \ldots, h_5 = I_3$;
- compute the middle convolution $\underline{\hat{g}}$ of $\underline{h}$ with respect to $\underline{\theta}$.

Since the middle convolution commutes with the mapping class group action on the character variety $\mathrm{Char}(0,5)$, the orbit of $[\underline{\hat{g}}] \in \mathrm{Char}(0,5)$ must be finite. This means that this procedure can be used to obtain potentially new such orbits, as we will demonstrate here.

5.1.2 List of orbits

The main result in this chapter lies with the following theorem; note however that this is still very much a work in progress.

Theorem G. *The classes associated to the following representations of the free group over four generators $\mathbf{F}_4 := \langle d_1, \ldots, d_4 \,|\, \emptyset \rangle$ in the character variety have finite orbits under the action of the mapping class group* $\mathrm{Mod}(0,5)$*:*

$$\rho_1 : d_1 \mapsto \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \; ;$$
$$\rho_2 : d_1 \mapsto \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ \frac{9}{2} & -\frac{1}{2} \end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \; ;$$
$$\rho_3 : d_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad d_2 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad d_3 \mapsto \begin{pmatrix} \frac{-15+i\sqrt{3}}{6} & -\frac{4+2i\sqrt{3}}{3} \\ 1 & \frac{9-i\sqrt{3}}{6} \end{pmatrix} \quad d_4 \mapsto \begin{pmatrix} \frac{9-i\sqrt{3}}{6} & \frac{1+i\sqrt{3}}{6} \\ -1 & \frac{3+i\sqrt{3}}{6} \end{pmatrix} .$$

Moreover, all three of these orbits are distinct.

This result is to be considered work in progress for the following reasons: first, we have not yet systematised our approach, meaning that any one of the finite subgroups of $SL_3(\mathbb{C})$ we made use of here could potentially yield multiple quintuplets of matrices $h_1, \ldots, h_5$ with double eigenvalues so that $h_1, \ldots, h_5 = I_3$, which could possibly lead to new finite orbits being obtained this way. Secondly, we have yet to explicitly describe the algebraic Garnier solution associated with the above orbits; we do however investigate a possible way to do so in one case in Section 5.3.2, but we only have partial results at the moment of writing.

Remark 5.1.1. *Recall from Section 2.1.2 that the class of a representation*

$$\rho : \mathbf{F}_4 = \langle d_1, \ldots, d_4 \,|\, \emptyset \rangle \to SL_2(\mathbb{C})$$

in the $SL_2(\mathbb{C})$–character variety $\mathrm{Char}(0,5)$ *of the five–punctured sphere is fully determined* *by the following:*

$$\begin{aligned}
t_1 &:= \mathrm{Tr}(\rho(d_1)),\ t_2 := \mathrm{Tr}(\rho(d_2)),\ t_3 := \mathrm{Tr}(\rho(d_3)),\\
t_4 &:= \mathrm{Tr}(\rho(d_4)),\ t_5 := \mathrm{Tr}(\rho(d_1d_2d_3d_4)),\\
r_1 &:= \mathrm{Tr}(\rho(d_1d_2)),\ r_2 := \mathrm{Tr}(\rho(d_1d_3)),\ r_3 := \mathrm{Tr}(\rho(d_1d_4)),\\
r_4 &:= \mathrm{Tr}(\rho(d_2d_3)),\ r_5 := \mathrm{Tr}(\rho(d_2d_4)),\ r_6 := \mathrm{Tr}(\rho(d_3d_4)),\\
r_7 &:= \mathrm{Tr}(\rho(d_1d_2d_3)),\ r_8 := \mathrm{Tr}(\rho(d_1d_2d_4)),\ r_9 := \mathrm{Tr}(\rho(d_1d_3d_4)),\\
r_{10} &:= \mathrm{Tr}(\rho(d_2d_3d_4)).
\end{aligned}$$

In these coordinates $(\underline{t}|\underline{r})$*, the point corresponding to the representations appearing in Theorem G are:*

- $[\rho_1] = (-2, 2, 2, 2, 2\,|\,-1, -1, -2, 2, 1, 1, 0, 0, 0, 0)$*;*
- $[\rho_2] = (-2, 2, 2, 2, 2\,|\,-1, -1, 0, 1, 2, 0, 2, 1, 1, -1)$*;*
- $[\rho_3] = (2, 2, -1, 2, 2\,|\,2, 0, 1, 0, 1, 2, 1, 0, 2, 2)$.

5.2 Effective Katz middle convolution

The aim of this paragraph is to prove Theorem G using explicit computer algebra methods. We will describe the algorithms we used and how we did in order to give the reader a complete picture.

5.2.1 Selecting free group representations

5.2.1.1 A first example

Let us begin by giving a simple example of a quintuplet of matrices in $SL_3(\mathbb{C})$ with double eigenvalues generating a free group; consider

$$h_1^1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},\ h_2^1 := \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ h_3^1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$h_4^1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } h_5^1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One easily checks that $h_1^1 \dots h_5^1 = I_3$ and that the subgroup they span is finite; moreover all admit -1 as a double eigenvalue. We will use this example to obtain the representation ρ_1 from Theorem G.

5.2.1.2 Finite relexion groups of $SL_3(\mathbb{C})$

Consider the following matrices in $GL_3(\mathbb{C})$:

$$r_1 := \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1+i\sqrt{7}}{4} \\ -\frac{1}{2} & \frac{1}{2} & \frac{-1+i\sqrt{7}}{4} \\ -\frac{1+i\sqrt{7}}{4} & -\frac{1+i\sqrt{7}}{4} & 0 \end{pmatrix}, \; r_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; r_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Also define:

$$u_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \;, u_2 := -\frac{i\sqrt{3}}{3} \begin{pmatrix} \omega & \omega^2 & \omega^2 \\ \omega^2 & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega \end{pmatrix}, \; u_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$\omega := \frac{1+\sqrt{5}}{2}.$$

From work by Shephard and Todd [54], we know that the groups spanned by the following triples of matrices are finite: (r_1, r_2, r_3) and (u_1, u_2, u_3); moreover they have (respectively) 336 and 182 elements.

5.2.1.3 Special quintuplets

The first step in the procedure is to select suitable quintuplets of matrices generating finite groups in $SL_3(\mathbb{C})$. We proceed as follows, using a simple brute–force method: we compute all the elements in the three groups mentioned above using the following algorithm.

```
BFG := proc(Identity,Gen, rec:=infinity)
local L,k,n0,n,i,j,M:
L := [Identity]:
k:=nops(Gen):
n0 := 0:
n := 1:
while n0<n and (n<rec) do
for i from 1 to k do
for j from 1 to n do
M:=map(x->factor(x),L[j].Gen[i]):
if not MatMember(M,L) then
L:=[op(L),M]: fi:
od: od:
n0 :=n :
n := nops(L):
od:
```

```
RETURN(L):
end proc:
```

The above algorithm (BruteForceGroup, or `BFG`) relies on the following entries: `Identity` is the identity element (used to initialise the induction), `Gen` is a list containing the generators of the group one wishes to compute and the optional parameter `rec` can be used to set a recursion limit (default is infinity). To use `BFG` to compute matrix groups one needs the `LinearAlgebra` `Maple` package and the following auxiliary function determining whether a matrix is a member of some list or not.

```
MatMember := proc(M,L)
local res, i:
res := false:
for i from 1 to nops(L) do
if Equal(M,L[i]) then
res := true: fi: od:
RETURN(res):
end proc:
```

Once this is done, we look for all elements in the group that have double eigenvalue ; this is done through the following procedure.

```
Doubles := proc(S)
local i,n,L:
L := []:
n:=nops(S):
for i from 1 to n do
if hasDouble(S[i]) then
L := [op(L),i]: fi:od:
RETURN(L): end proc:
```

Here, the entry must be the indexed list of the eigenvalues of elements of the group; more precisely, if `G` is the list computed using `BFG`, verb+S+ must be the list

```
S:=[seq(Eigenvalues(G[i]), i=1..nops(G))]:
```

we also make use of the following auxiliary function to find out if some list has repeated elements.

```
hasDouble := proc(L)
local i,j,res, n:
n:= nops(L):
res := false:
for j from 1 to n do
for i from j+1 to n do
```

```
if L[i] = L[j] then
res := true: fi:od:od:
RETURN(res):
end proc:
```

Once we are in possession of the list returned by `Doubles`, which contains the indexes i such that `G[i]` has double eigenvalue, we make use of one last `Maple` procedure to find quintuplets of such matrices whose product is a scalar matrix.

```
Quint := proc(ind,G,i:=1)
local i1, i2, i3, i4, i5, n, L, P, ok:
L:=[]:
ok :=0:
n:=nops(ind):
while evalb(ok<i) do
for i1 from 2 to n do
for i2 from i1+1 to n do
for i3 from i2+1 to n do
for i4 from i3+1 to n do
for i5 from i4+1 to n do
if evalb(ok<i) then
P := map(x->factor(x),G[ind[i1]].G[ind[i2]].G[ind[i3]].G[ind[
    i4]].G[ind[i5]]):
if IsMatrixShape(P,scalar)  then
L := [op(L), [ind[i1],ind[i2],ind[i3],ind[i4],ind[i5]]]:ok:=ok
    +1: print(ok,[ind[i1],ind[i2],ind[i3],ind[i4],ind[i5]]):
    fi:fi:
od:od:od:od:od:od:
RETURN(L):
end proc:
```

Here the entries are as follows: `ind` is the list of suitable indexes returned by `Doubles`, `G` is the list of all elements in the group returned by `BFG` and `i` (by default equal to 1) is the number of quintuplets one wishes the algorithm to return before stopping.

5.2.1.4 Explicit quintuplets

To obtain the representations ρ_2 and ρ_3 we run through the above procedure once with (in that order) the groups generated by (r_1, r_2, r_3) and (u_1, u_2, u_3). This gives us the following quintuplets of matrices in $GL_3(\mathbb{C})$:

$$h_1^2 := \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{-1+i\sqrt{7}}{4} \\ -\frac{1}{2} & \frac{1}{2} & \frac{-1+i\sqrt{7}}{4} \\ -\frac{1+i\sqrt{7}}{4} & -\frac{1+i\sqrt{7}}{4} & 0 \end{pmatrix},\ h_2^2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ h_3^2 := \begin{pmatrix} \frac{1}{2} & \frac{-1+i\sqrt{7}}{4} & -\frac{1}{2} \\ -\frac{1+i\sqrt{7}}{4} & 0 & -\frac{1+i\sqrt{7}}{4} \\ -\frac{1}{2} & \frac{-1+i\sqrt{7}}{4} & \frac{1}{2} \end{pmatrix},$$

$$h_4^2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},\ h_5^2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and

$$h_1^3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1+i\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ h_2^3 := \begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & & \\ 0 & 0 & -\frac{1+i\sqrt{3}}{2} & 0 \end{pmatrix},\ h_3^3 := \begin{pmatrix} \frac{3+i\sqrt{3}}{6} & \frac{-3+i\sqrt{3}}{6} & \frac{-3+i\sqrt{3}}{6} \\ \frac{-3+i\sqrt{3}}{6} & \frac{3+i\sqrt{3}}{6} & \frac{-3+i\sqrt{3}}{6} \\ \frac{-3+i\sqrt{3}}{6} & \frac{-3+i\sqrt{3}}{6}2 & \frac{3+i\sqrt{3}}{6} \end{pmatrix},$$

$$h_4^3 := \begin{pmatrix} \frac{3-i\sqrt{3}}{6} & -\frac{3+i\sqrt{3}}{6} & -\frac{3+i\sqrt{3}}{6} \\ -\frac{3+i\sqrt{3}}{6} & \frac{3-i\sqrt{3}}{6} & -\frac{3+i\sqrt{3}}{6} \\ -\frac{3+i\sqrt{3}}{6} & -\frac{3+i\sqrt{3}}{6} & \frac{3-i\sqrt{3}}{6} \end{pmatrix},\ h_5^3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{-1+i\sqrt{3}}{2} \end{pmatrix}.$$

Note that for $i \in [3]$, we have $h_1^i \dots h_5^i = I_3$.

5.2.2 Middle convolution algorithm

5.2.2.1 Normalisation

Recall from Section I2 in the introduction that to use Katz's middle convolution algorithm we need a n–uple of matrices $\underline{g} = (g_1, \dots, g_r)$ such that $g_1 \dots g_r$ is the identity matrix and so that the following two conditions are met:

(CIK1) the matrix g_r commutes to all g_i, for $i \in [r-1]$;

(CIK2) 1 is not an eigenvalue of the matrix g_r.

In our case, we follow the steps given in Section I2.2. Let us take $r = 6$ and, in order to obtain rank two representations, we need to ensure that $g_1, \dots, g_5$ have 1 as a double eigenvalue. This leads us to set:

- $g_i^1 = -h_i^1$ for $i \in [5]$ and $g_6^1 := -I_3$;
- $g_i^2 = -h_i^2$ for $i \in [5]$ and $g_6^2 := -I_3$;
- $g_i^3 = -h_i^3$ for $i \in [4]$, $g_5^3 := -\frac{2}{1-i\sqrt{3}} h_5^3$ and $g_6^2 := \frac{-1+i\sqrt{3}}{2} I_3$.

One checks that this gives us three sextuplets $\underline{g}^1, \underline{g}^2, \underline{g}^3$ of matrices in $GL_3(\mathbb{C})$ satisfying conditions **(CIK1)** and **(CIK2)**; moreover, one also gets that for all $j \in [3]$, $g_1^j \dots g_6^j = I_3$ and

$$\sum_{i=1}^{5} \mathrm{rank}(I_d - g_i^j) = 5 \ .$$

5.2.2.2 The algorithm

Now we describe the algorithm we use to compute the actual Katz's middle convolution of the sextuplets $\underline{g}^1, \underline{g}^2$ and $\underline{g}^3$. This draws heavily on Völklein's formalism, as detailed in Section I2; as such we will use the same notations here than there. This procedure requires the following set–up commands.

```
with(LinearAlgebra):
o:=ZeroMatrix(3):
Id := IdentityMatrix(3):
```

Assume that we start with `g1`, `g2`, `g3`, `g4`, `g5` and `g6` six three by three matrices making up one of the sextuplets $\underline{g}^j$. Start by setting up the invariant space U_ρ pertaining to the group representation $\rho : \mathbf{F}_5 \to GL_3(\mathbb{C})$ associated with $\underline{g}^j$. To do so, we first compute the vector space $\mathrm{Im}(I_3 - g_1^j) \times \cdots \times \mathrm{Im}(I_3 - g_6^j)$:

```
ROW:=op([RowSpace(g1-Id),  RowSpace(g2-Id), RowSpace(g3-Id),
    RowSpace(g4-Id), RowSpace(g5-Id), RowSpace(g6-Id) ]);
vect :=simplify(Vector([op(convert(y1*ROW[1][1],list)), op(
    convert(y2*ROW[2][1],list)) ,op(convert(y3*ROW[3][1],list))
    ,op(convert(y4*ROW[4][1],list)),op(convert(y5*ROW[5][1],
    list)),x6,y6,z6]));
```

and set up the matrix `M` corresponding to $\varphi(\rho)$.

```
M := map(x->factor(x),Matrix([[ Id ],[MatrixInverse(g1)], [
    MatrixInverse(g1.g2)],[MatrixInverse(g1.g2.g3)],[
    MatrixInverse(g1.g2.g3.g4)],[MatrixInverse(g1.g2.g3.g4.g5)
    ]]));
```

The invariant set U_ρ is then equal to the intersection between $\mathrm{Im}(I_3 - g_1^j) \times \cdots \times \mathrm{Im}(I_3 - g_6^j)$ and the kernel of `M`; it is a vector space of dimension (*cf.* equation (E10)

$$\dim(U_\rho) = \sum_{i=1}^{5} \mathrm{rank}(I_d - g_i) = 5 \tag{E5.1}$$

of which we can explicitly compute a basis, say `e1`, ..., `e5`, using `Maple`. We know use the explicit formulas given in Section I2 to give the matrices `F1`, ..., `F5` corresponding to the action of the pure braids

$$F_i = (\sigma_{r-1} \dots \sigma_{i+1})\sigma_i^2(\sigma_{r-1} \dots \sigma_{i+1})^{-1}$$

on U_ρ.

```
F1 := Transpose(Matrix([[ g6 , g1.(Id-g6).(Id-MatrixInverse(g2)) , g1.(Id-g6).(Id-MatrixInverse(g3)) , g1.(Id-g6).(Id-MatrixInverse(g4)) , g1.(Id-g6).(Id-MatrixInverse(g5)) , g1.(Id-g6) ],
        [ o , Id , o , o , o , o ],
        [ o , o , Id , o , o , o ],
        [ o , o , o , Id , o , o ],
        [ o , o , o , o , Id , o ],
        [ g6.(MatrixInverse(g1)-Id) , g6.(g1-Id).(Id-MatrixInverse(g2)) , g6.(g1-Id).(Id-MatrixInverse(g3)) , g6.(g1-Id).(Id-MatrixInverse(g4)) , g6.(g1-Id).(Id-MatrixInverse(g5)) , Id-g6+g1.g6 ]])):
F2 := Transpose(Matrix([[ Id,o,o,o,o,o ],
        [ o, g6 , g2.(Id-g6).(Id-MatrixInverse(g3)) , g2.(Id-g6).(Id-MatrixInverse(g4)) , g2.(Id-g6).(Id-MatrixInverse(g5)) ,  g2.(Id-g6)],
        [ o , o , Id , o , o , o ],
        [ o , o , o , Id , o , o ],
        [ o , o , o , o , Id , o ],
        [ o, g6.(MatrixInverse(g2)-Id) , g6.(g2-Id).(Id-MatrixInverse(g3)) , g6.(g2-Id).(Id-MatrixInverse(g4)) , g6.(g2-Id).(Id-MatrixInverse(g5)) , Id-g6+g2.g6 ]])):

F3 := Transpose(Matrix([[ Id,o,o,o,o,o ],
        [ o , Id , o , o , o , o ],
        [ o,o, g6 , g3.(Id-g6).(Id-MatrixInverse(g4)) , g3.(Id-g6).(Id-MatrixInverse(g5)) ,   g3.(Id-g6) ],
        [ o , o , o , Id , o , o ],
        [ o , o , o , o , Id , o ],
        [ o, o, g6.(MatrixInverse(g3)-Id) , g6.(g3-Id).(Id-MatrixInverse(g4)) , g6.(g3-Id).(Id-MatrixInverse(g5)) ,  Id-g6+g3.g6 ]])):
F4 := Transpose(Matrix([[ Id,o,o,o,o,o ],
        [ o , Id , o , o , o , o ],
        [ o , o , Id , o , o , o ],
```

```
        [ o,o, o,g6 , g4.(Id-g6).(Id-MatrixInverse(g5)) ,
           g4.(Id-g6) ],
        [ o , o , o , o , Id , o ],
        [ o, o, o, g6.(MatrixInverse(g4)-Id) , g6.(g4-Id).(Id-
           MatrixInverse(g5)) , Id-g6+g4.g6 ]])):
F5 := Transpose(Matrix([[ Id,o,o,o,o,o ],
        [ o , Id , o , o , o , o ],
        [ o , o , Id , o , o , o ],
        [   o , o , o , Id , o , o  ],
        [ o,o, o, o, g6     ,g5.(Id-g6) ],
        [ o, o, o, o, g6.(MatrixInverse(g5)-Id) ,   Id-g6+g5.
           g6 ]])):
```

We then turn our attention to the subspace $V_\rho \subset U_\rho$ made up of the trivial affine extensions of ρ, namely the image of the mapping $\psi(\rho)$ that we represent by the matrix `N` below.

```
N:=map(x->factor(x),Matrix([[ Id - MatrixInverse(g1), Id -
   MatrixInverse(g2), Id - MatrixInverse(g3), Id -
   MatrixInverse(g4), Id - MatrixInverse(g5),Id -
   MatrixInverse(g6) ]]));
```

In order to explicitly compute the action of the braids F_i on the quotient $E_\rho = U_\rho/V_\rho$ we need to find an adequate basis of U_ρ.

```
f1 := simplify(Transpose(Row(N,1))):
f2 :=simplify(Transpose(Row(N,2))):
f3 :=simplify(Transpose(Row(N,3))):
```

Assume for the sake of simplicity that the rank of the family (`e1`,`e2`,`f1`,`f2`,`f3`) is five (up to changing one or more of the `e`i); now we only need to compute the components of the vectors F_j(`e`i) along `e1` and `e2` to obtain the matrices $\tilde{g}_i^j \in GL_2(\mathbb{C})$ corresponding to the middle convolution of $\underline{g}^j$.

```
sol11 := solve(convert(map(x->factor(x),F1.e1 - (x1*e1+x2*e2+
   x3*f1+x4*f2+x5*f3)),set),{x1,x2,x3,x4,x5}):
sol12 := solve(convert(map(x->factor(x),F1.e2 - (x1*e1+x2*e2+
   x3*f1+x4*f2+x5*f3)),set),{x1,x2,x3,x4,x5}):
h1 :=simplify(Transpose(Matrix([[ eval(x1,sol11) , eval(x1,
   sol12)   ],
        [ eval(x2,sol11) , eval(x2,sol12)  ]])));

sol21 := solve(convert(map(x->factor(x),F2.e1 - (x1*e1+x2*e2+
   x3*f1+x4*f2+x5*f3)),set),{x1,x2,x3,x4,x5}):
```

```
sol22 := solve(convert(map(x->factor(x),F2.e2 - (x1*e1+x2*e2+x3*f1+x4*f2+x5*f3)),set),{x1,x2,x3,x4,x5}):
h2 :=simplify(Transpose(Matrix([[ eval(x1,sol21) , eval(x1,sol22)  ],
        [ eval(x2,sol21) , eval(x2,sol22)  ]])));

sol31 := solve(convert(map(x->factor(x),F3.e1 - (x1*e1+x2*e2+x3*f1+x4*f2+x5*f3)),set),{x1,x2,x3,x4,x5}):
sol32 := solve(convert(map(x->factor(x),F3.e2 - (x1*e1+x2*e2+x3*f1+x4*f2+x5*f3)),set),{x1,x2,x3,x4,x5}):
h3 :=simplify(Transpose(Matrix([[ eval(x1,sol31) , eval(x1,sol32)  ],
        [ eval(x2,sol31) , eval(x2,sol32)  ]])));

sol41 := solve(convert(map(x->factor(x),F4.e1 - (x1*e1+x2*e2+x3*f1+x4*f2+x5*f3)),set),{x1,x2,x3,x4,x5}):
sol42 := solve(convert(map(x->factor(x),F4.e2 - (x1*e1+x2*e2+x3*f1+x4*f2+x5*f3)),set),{x1,x2,x3,x4,x5}):
h4 :=simplify(Transpose(Matrix([[ eval(x1,sol41) , eval(x1,sol42)  ],
        [ eval(x2,sol41) , eval(x2,sol42)  ]])));

sol51 := solve(convert(map(x->factor(x),F5.e1 - (x1*e1+x2*e2+x3*f1+x4*f2+x5*f3)),set),{x1,x2,x3,x4,x5}):
sol52 := solve(convert(map(x->factor(x),F5.e2 - (x1*e1+x2*e2+x3*f1+x4*f2+x5*f3)),set),{x1,x2,x3,x4,x5}):
h5 :=simplify(Transpose(Matrix([[ eval(x1,sol51) , eval(x1,sol52)  ],
        [ eval(x2,sol51) , eval(x2,sol52)  ]])));
```

By normalising these new two by two matrices as described in Section I2.2, we obtain the representations ρ_1, ρ_2 and ρ_3 from (respectively) $\underline{g}^1$, $\underline{g}^2$ and $\underline{g}^3$. Since these are the middle convolution of representations with finite image, the equivariance of the latter under the mapping class group action implies that the points $[\rho_1]$, $[\rho_2]$ and $[\rho_3]$ in the character variety $\mathrm{Char}(0,5)$ have finite orbit under the aforementioned group action.

5.3 Further study of the mapping class group orbits

In this paragraph, we concern ourselves with furthering our understanding of the mapping class group orbits of the points $[\rho_1]$, $[\rho_2]$ and $[\rho_3]$ in the character variety $\mathrm{Char}(0,5)$ corresponding to the representations appearing in Theorem G. More precisely, we explicitly compute the aforementioned orbits before looking at the different ways to have one of them degenerate onto algebraic solutions of the Painlevé VI equation. Note that this is to be considered a work in progress and is to be treated more as an invitation to further study than as a meaningful result.

5.3.1 Explicit mapping class group orbits

In the following section, we compute the orbits of the points $[\rho_1]$, $[\rho_2]$ and $[\rho_3]$ in the character variety $\mathrm{Char}(0,5)$ under the action of the pure mapping class group $\mathrm{PMod}(0,5)$ by using the algorithm described in Appendix A.

Before we do so, let us remark that the mapping class group orbits of the points $[\rho_1]$, $[\rho_2]$ and $[\rho_3]$ under the complete mapping class group $\mathrm{Mod}(0,5)$ are pairwise distinct. Indeed, using the aforementioned algorithm, we get that they are of pairwise distinct sizes and so cannot be equal.

5.3.1.1 Mapping class group orbit of $[\rho_1]$

The orbit of the first representation appearing in Theorem G under the pure mapping class group is of size 12 and is made up of the following points in the character variety:

1. $(-2,2,2,2,2\,|-1,-1,-2,2,1,1,0,0,0,0)$;
2. $(-2,2,2,2,2\,|-2,-1,-1,1,1,2,0,0,0,0)$;
3. $(-2,2,2,2,2\,|-1,-2,-1,1,1,1,0,2,0,0)$;
4. $(-2,2,2,2,2\,|-1,-1,-1,1,-2,1,0,2,0,-2)$;
5. $(-2,2,2,2,2\,|-1,-2,-1,1,1,1,0,0,0,-2)$;
6. $(-2,2,2,2,2\,|-1,2,-1,1,1,1,2,0,2,0)$;
7. $(-2,2,2,2,2\,|-1,-1,-1,1,2,1,2,0,0,0)$;
8. $(-2,2,2,2,2\,|-1,-1,-1,1,2,1,0,0,2,0)$;
9. $(-2,2,2,2,2\,|2,-1,-1,1,1,2,2,2,0,0)$;
10. $(-2,2,2,2,2\,|-1,-1,2,2,1,1,0,2,2,0)$;
11. $(-2,2,2,2,2\,|-1,-1,-2,-2,1,1,2,0,0,-2)$;
12. $(-2,2,2,2,2\,|-2,-1,-1,1,1,-2,0,0,2,-2)$.

5.3.1.2 Mapping class group orbit of $[\rho_2]$

This orbit is also of size 12, but is distinct from that of $[\rho_1]$. Its elements are listed below:

1. $(-2,2,2,2,2\,|-1,-1,0,1,2,0,2,1,1,-1)$;
2. $(-2,2,2,2,2\,|-1,-1,0,1,0,2,0,1,1,-1)$;
3. $(-2,2,2,2,2\,|-1,-1,-2,1,0,0,0,1,1,-1)$;
4. $(-2,2,2,2,2\,|\,2,-1,0,1,0,2,2,2,1,-1)$;
5. $(-2,2,2,2,2\,|-1,2,0,1,0,0,2,1,2,-1)$;
6. $(-2,2,2,2,2\,|-1,-1,-2,-2,0,0,2,1,1,-2)$;
7. $(-2,2,2,2,2\,|-2,-1,-2,1,2,0,0,-2,1,-1)$;
8. $(-2,2,2,2,2\,|-1,-2,0,1,2,0,0,1,2,-1)$;
9. $(-2,2,2,2,2\,|-1,-1,0,2,2,2,0,1,1,2)$;
10. $(-2,2,2,2,2\,|-1,-1,0,2,0,0,0,1,1,-2)$;
11. $(-2,2,2,2,2\,|-2,-1,0,1,0,0,0,2,1,-1)$;
12. $(-2,2,2,2,2\,|-1,-2,-2,1,0,2,0,1,-2,-1)$.

5.3.1.3 Mapping class group orbit of $[\rho_3]$

This last orbit is of size 9, made up of the following points in the character variety of the five punctured sphere:

1. $(2,2,-1,2,2\,|\,2,0,1,0,1,2,1,0,2,2)$;
2. $(2,2,-1,2,2\,|\,1,0,1,0,2,0,1,0,2,1)$;
3. $(2,2,-1,2,2\,|\,1,0,1,0,1,0,1,0,1,2)$;
4. $(2,2,-1,2,2\,|\,1,2,1,0,1,0,2,0,2,1)$;
5. $(2,2,-1,2,2\,|\,1,0,2,2,1,0,2,0,1,2)$;
6. $(2,2,-1,2,2\,|\,2,0,1,0,1,0,1,0,1,1)$;
7. $(2,2,-1,2,2\,|\,1,0,2,0,1,0,1,0,1,1)$;
8. $(2,2,-1,2,2\,|\,2,0,2,0,2,0,1,2,1,1)$;
9. $(2,2,-1,2,2\,|\,1,0,1,0,2,0,2,0,1,1)$.

5.3.2 Regression to Painlevé VI

In this paragraph, we take a closer look at the orbit of the point

$$[\rho_3] = (2, 2, -1, 2, 2 \,|\, 2, 0, 1, 0, 1, 2, 1, 0, 2, 2) \in \mathrm{Char}(0, 5)$$

under the action of the pure mapping class group PMod(0, 5). The idea here is the following: since the orbit of $[\rho_3]$ under the mapping class group action is of size 9, the corresponding algebraic Garnier solution must be a nine–branched multivalued function over $\mathbb{P}^1 \times \mathbb{P}^1$ (with coordinates t_1 and t_2) and the action of PMod(0, 5) permutes these nine branches. This means that if one was to associate an actual isomonodromic deformation to $[\rho_3]$, one would have to do so on a 9 : 1 branched cover over $\mathbb{P}^1 \times \mathbb{P}^1$, so the upshot would be to fully understand the latter.

One angle we explored during this thesis was to try and understand how such an isomonodromic deformation would behave should two of its singular points conflate. More precisely, imagine that we have an isomonodromic deformation $(\nabla_{t_1,t_2})_{t_1,t_2}$ of the five punctured sphere with monodromy ρ_3; this implies in particular that for any pair (t_1, t_2) of distinct points in $\mathbb{C}^* \setminus \{0, 1\}$, ∇_{t_1,t_2} is a logarithmic flat connection over the punctured sphere $\mathbb{P}^1 \setminus \{0, 1, t_1, t_2, \infty\}$. If one looks at the (formal) limit of this family of connections when (for example) t_1 goes to 0, one gets an isomonodromic deformation over the four punctured sphere and thus obtains an algebraic solution of the Painlevé VI equation. Since those have been classified [42], we can give a complete description of those "regressions".

5.3.2.1 Excerpt from the classification of algebraic Painlevé VI solutions

What follows is taken directly from [42]; we give a brief description of the algebraic solutions of the Painlevé VI that we will need later on. Recall that the following algebraic coordinates characterise the point in Char(0, 4) associated with some quadruplet of matrices $(M_1, \dots, M_4) \in SL_2(\mathbb{C})^4$ whose product is the identity (see Section I1.3):

$$a := \mathrm{Tr}(M_1), \quad b := \mathrm{Tr}(M_2), \quad c := \mathrm{Tr}(M_3), \quad d := \mathrm{Tr}(M_4),$$
$$x := \mathrm{Tr}(M_1M_2), \quad y := \mathrm{Tr}(M_2M_3), \quad z := -\mathrm{Tr}(M_1M_3).$$

However, for the sake of convenience, we will use the same coordinates than Tykhyy and Lisovyy in their paper [42]:

$$\omega_X := \mathrm{Tr}(M_3)\mathrm{Tr}(M4) + \mathrm{Tr}(M_1)\mathrm{Tr}(M_2), \quad \omega_Y := \mathrm{Tr}(M_2)\mathrm{Tr}(M4) + \mathrm{Tr}(M_1)\mathrm{Tr}(M_3),$$
$$\omega_Z := \mathrm{Tr}(M_1)\mathrm{Tr}(M4) + \mathrm{Tr}(M_2)\mathrm{Tr}(M_3),$$
$$X := \mathrm{Tr}(M_1M_2), \quad Y := \mathrm{Tr}(M_1M_3), \quad Z := -\mathrm{Tr}(M_2M_3).$$

Note that ω_X, ω_Y and ω_Z are invariant under the action of PMod(0, 4). The classification by Lisovyy and Tykhyy of the algebraic solutions of the sixth Painlevé equation tells us that the only non–rigid families of solutions are the following (up to Okamoto symmetries):

	$t_1 = 0$	$t_1 = 1$	$t_1 = \infty$	$t_2 = 0$	$t_2 = 1$	$t_2 = \infty$	$t_1 = t_2$
1	IV	IV	IV	II	II	II	I
2	IV	IV	IV	II	III	II	IV
3	IV	IV	IV	II	II	II	IV
4	I	IV	IV	II	II	III	IV
5	IV	I	IV	III	II	III	IV
6	IV	IV	I	II	II	II	IV
7	IV	IV	IV	III	II	II	IV
8	IV	IV	IV	III	III	II	IV
9	IV	IV	IV	II	III	III	IV

Table 5.1: "Painlevé regressions" for ρ_3.

- *Type I solutions:* those are the solutions who are also fixed points under the action of PMod$(0,4)$; they satisfy the following equations:

$$\omega_X = 2X + YZ, \quad \omega_Y = 2Y + XZ, \quad \omega_Z = 2Z + XY\ .$$

- *Type II solutions:* up to Okamoto symmetries, those are the solutions whose orbit is made of two points $(X, Y, Z) = (a, 0, 0)$ and $(b, 0, 0)$ with parameters $\omega_X = a + b$ and $\omega_Y = \omega_Z = 0$; moreover X' and X'' are free parameters of the form $a = 2\cos(2\pi\theta)$ and $b = -2\cos(2\pi\vartheta)$.

- *Type III solutions:* those are the solutions whose orbit is made up of three points $(X, Y, Z) = (1, 0, 0)$, $(1, \omega, 0)$ and $(1, 0, \omega)$, for some $\omega \in \mathbb{C}$, with parameters $\omega_X = 2$ and $\omega_Y = \omega_Z = \omega$.

- *Type IV solutions:* those are the solutions whose orbit is made up of four points $(X, Y, Z) = (1, 1, 1)$, $(\omega - 2, 1, 1)$, $(1, \omega - 2, 1)$ and $(1, 1, \omega - 2)$, for some $\omega \in \mathbb{C}$, with parameters $\omega_X = \omega_Y = \omega_Z = \omega$.

5.3.2.2 Explicit regressions

We now have the necessary tools to give the "Painlevé regressions" of the algebraic Garnier solution associated with the representation ρ_3, which we list in Table 5.1; the pure mapping class group orbits having been computed using the method described in Appendix A. For each point in the orbit of $[\rho_3]$ one has seven possible restrictions, namely $t_1, t_2 = 0, 1, \infty$ and $t_1 = t_2$; in each case we give the type of the associated algebraic Painlevé VI solution. All type II (resp. III and IV) solutions appearing in this table have parameters $X' = 2$ and $X'' = 1$ (resp. $\omega = 4$) and all type I solutions correspond to the point $(X, Y, Z) = (2, 2, 2)$.

Chapter 6

Virtual elliptic middle convolution

In this chapter we investigate a possible extension of Katz's middle convolution to representations of the fundamental group of a twice punctured torus. This is done mainly by extending Völklein's formalism [57]; however we unfortunately only manage to obtain new representations of some finite index subgroup of the original fundamental group.

6.1 Framework and main result

Let x_0, p_1, p_2 and p_3 be four pairwise distinct points in the complex torus $\mathbb{T}^2 := \mathbb{C}/\mathbb{Z}^2$, and let Γ_2 (resp. Γ_3) be the fundamental group $\pi_1(\mathbb{T}^2\backslash\{p_1, p_2\}, x_0)$ (resp. $\pi_1(\mathbb{T}^2\backslash\{p_1, p_2, p_3\}, x_0)$). Then there is a natural "outer action" of Γ_2 on Γ_3, i.e a group homomorphism

$$\Gamma_2 \to \mathrm{Out}(\Gamma_3)$$

defined as follows. It is well known that if one considers the loops drawn in Fig. 6.1, then the groups Γ_i admit the following presentations:

$$\Gamma_2 = \langle \gamma_1, \gamma_2, \alpha, \beta \,|\, \gamma_1\gamma_2[\alpha, \beta] = 1\rangle \quad \text{and} \quad \Gamma_3 = \langle \gamma_1, \gamma_2, \gamma_3, \alpha, \beta \,|\, \gamma_1\gamma_2\gamma_3[\alpha, \beta] = 1\rangle \ .$$

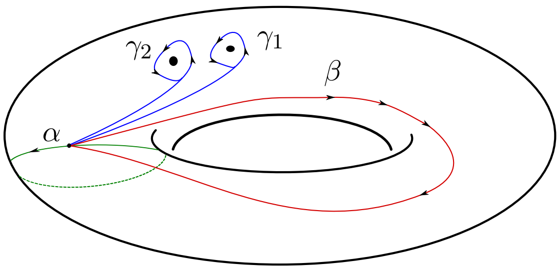

Figure 6.1: Generators of Γ_2.

Then for any loop $\delta \in \pi_1(\mathbb{T}^2 \setminus \{p_1, p_2\}, p_3)$ (note the change of base point here), continuously deform the loops γ_1, γ_2, γ_3, α and β by moving the puncture p_3 along δ. Using the canonical isomorphism between Γ_2 and $\pi_1(\mathbb{T}^2 \setminus \{p_1, p_2\}, p_3)$, one thus obtains the following formulas to describe this "outer action".

	γ_1	γ_2	γ_3
γ_1	$(\gamma_1\gamma_3)\gamma_1(\gamma_1\gamma_3)^{-1}$	$[\gamma_1,\gamma_3]\gamma_2[\gamma_3,\gamma_1]$	$\gamma_1\gamma_3\gamma_1^{-1}$
γ_2	γ_1	$(\gamma_2\gamma_3)\gamma_2(\gamma_2\gamma_3)^{-1}$	$\gamma_2\gamma_3\gamma_2^{-1}$
α	$\gamma_3^{-1}\gamma_1\gamma_3$	$\gamma_3^{-1}\gamma_2\gamma_3$	$\alpha\gamma_3\alpha^{-1}$
β	$(\beta\gamma_3\beta^{-1})\gamma_1(\beta\gamma_3\beta^{-1})^{-1}$	$(\beta\gamma_3\beta^{-1})\gamma_2(\beta\gamma_3\beta^{-1})^{-1}$	$\beta\gamma_3\beta^{-1}$
$\gamma_1\gamma_2$	$[\beta,\alpha](\gamma_1\gamma_2)^{-1}\gamma_1(\gamma_1\gamma_2)[\alpha,\beta]$	$[\beta,\alpha](\gamma_1\gamma_2)^{-1}\gamma_2(\gamma_1\gamma_2)[\alpha,\beta]$	$(\gamma_1\gamma_2)\gamma_3(\gamma_1\gamma_2)^{-1}$
α^{-1}	$(\alpha^{-1}\gamma_3\alpha)\gamma_1(\alpha^{-1}\gamma_3\alpha)^{-1}$	$(\alpha^{-1}\gamma_3\alpha)\gamma_2(\alpha^{-1}\gamma_3\alpha)^{-1}$	$\alpha^{-1}\gamma_3\alpha$
β^{-1}	$\gamma_3^{-1}\gamma_1\gamma_3$	$\gamma_3^{-1}\gamma_2\gamma_3$	$\beta^{-1}\gamma_3\beta$

	α	β
γ_1	α	β
γ_2	α	β
α	α	$\gamma_3^{-1}\beta$
β	$\gamma_3\alpha$	β
$\gamma_1\gamma_2$	α	β
α^{-1}	α	$(\alpha^{-1}\gamma_3\alpha)\beta$
β^{-1}	$(\beta^{-1}\gamma_3^{-1}\beta)\alpha$	β

Note that the element $\gamma_1\gamma_2[\alpha,\beta]$ "acts" as the conjugacy by $\alpha^{-1}\beta^{-1}\gamma_3\beta\alpha$.

Theorem H. *For any $r \geq 2$, there exists an explicit middle convolution (see Definition I2.1) between the twice–punctured torus and the $2r$ times punctured torus, relatively to the fundamental group of the two punctured torus.*

This is a weaker analogue to Katz's middle convolution as we only obtain a representation of some finite index subgroup.

6.2 Elliptic middle convolution

We begin by proving Theorem H through giving a complete description of our procedure.

6.2.1 *Ad–hoc* fibre bundle

Fix $g_1, g_2, g_3, A, B \in GL_d(\mathbb{C})$ such that:

(CIE1) $g_1g_2g_3[A,B] = I_d$, where $[A,B] = ABA^{-1}B^{-1}$;

(CIE2) $g_3 = c_3 I_d$ where $c_3 \in \mathbb{C}^* \setminus \{1\}$ is some r–th root of unity,

and define the following map:

$$\begin{aligned}\varphi : \mathbb{C}^2 &\to \mathrm{Mat}_{5d,d}(\mathbb{C})\\ (x,y) &\mapsto \begin{pmatrix} I_d & g_1 & g_1g_2 & [B,A] - yB & xBAB^{-1} - I_d \end{pmatrix}.\end{aligned}$$

The above defines a morphism of vector bundles

$$\begin{tikzcd} \mathbb{C}^2 \times X \arrow[r, "\varphi"] \arrow[d] & \mathbb{C}^2 \times \mathbb{C}^d \arrow[dl] \\ \mathbb{C}^2 & \end{tikzcd}$$

where $X := \mathrm{Im}(I_d - g_1) \times \mathrm{Im}(I_d - g_2) \times \mathbb{C}^{3d}$; note that for all $x, y \in \mathbb{C}^2$ one has

$$\mathrm{rank}(\varphi(x,y)) = d \,.$$

This means that one can consider the rank ℓ kernel bundle $U := \mathrm{Ker}(\varphi) \to \mathbb{C}^2$, with

$$\ell := \mathrm{rank}(I_d - g_1) + \mathrm{rank}(I_d - g_2) + 2d \,.$$

6.2.2 Affine group and action on affine representations

Recall that the d–dimensional affine group on a field k is defined as the semi–direct product $GA_d(k) := k^d \rtimes GL_d(k)$, where if $M \in GL_d(k)$ and $u \in k^d$ we denote by (u, M) the associated element in $GA_d(k)$ and set:

$$\forall M, N \in GL_d(k),\ \forall u, v \in k^d, \quad (u, M) \cdot (v, N) := (u + Mv, MN) \,.$$

There is a natural group homomorphism between $GA_d(k)$ and $GL_{d+1}(k)$ given by:

$$(u, M) \mapsto \begin{pmatrix} M & u \\ 0 & I_d \end{pmatrix} \,.$$

Let $\rho : \Gamma_3 \to GA_d(k)$ be a group morphism. Then the above "outer action" of Γ_2 on Γ_3 gives rise to a group homomorphism $\Gamma_2 \to \mathrm{Out}(\mathrm{Im}(\rho)) \le GA_d(k)$.

Let $(x, y) \in \mathbb{C}^2$ and $\underline{u} \in U_{(x,y)}$; one has a natural group morphism $\rho : \Gamma_3 \to GA_d(\mathbb{C})$ given by $\forall i \in [3]$, $\rho(\gamma_i) := (u_i, g_i)$, $\rho(\alpha) := (u_A, xA)$ and $\rho(\beta) := (u_B, yB)$. Thus, the "outer action" of Γ_2 on Γ_3 described earlier gives us four mappings F_1, F_2, F_A and F_B, associated with the loops γ_1, γ_2, α and β in Γ_2, that can be described as follows.

- F_1 and F_2 act trivially on the base, so they can be seen as linear automorphisms of the fibre $U_{(x,y)}$ acting as the right multiplication by the following matrices:

$$F_1 := {}^t\begin{pmatrix} I_d + g_3 g_1 - g_1 & (g_3 - I_d)(g_2 - I_d) & I_d - g_3 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ g_1(I_d - g_1) & (g_1 - I_d)(I_d - g_2) & g_1 & 0 & 0 \\ 0 & 0 & 0 & I_d & 0 \\ 0 & 0 & 0 & 0 & I_d \end{pmatrix},$$

$$F_2 := {}^t\begin{pmatrix} I_d & 0 & 0 & 0 & 0 \\ 0 & I_d + g_2 g_3 - g_2 & (I_d - g_3) & 0 & 0 \\ 0 & g_2(I_d - g_2) & g_2 & 0 & 0 \\ 0 & 0 & 0 & I_d & 0 \\ 0 & 0 & 0 & 0 & I_d \end{pmatrix}.$$

- F_A acts as $(x, y) \to \left(x, \frac{y}{c_3}\right)$ on the base and as a linear isomorphism between fibres $U_{(x,y)}$ and $U_{(x,c_3^{-1}y)}$ given by the right multiplication by the matrix:

$$F_A := {}^t\begin{pmatrix} g_3^{-1} & 0 & 0 & 0 & 0 \\ 0 & g_3^{-1} & 0 & 0 & 0 \\ g_3^{-1}(g_1 - I_d) & g_3^{-1}(g_2 - I_d) & xA & 0 & -g_3^{-1} \\ 0 & 0 & (I_d - g_3) & I_d & 0 \\ 0 & 0 & 0 & 0 & g_3^{-1} \end{pmatrix}.$$

- Similarly, F_B acts as $(x, y) \to (x, c_3 y)$ on the base and as a linear isomorphism between fibres $U_{(x,y)}$ and $U_{(x,c_3y)}$ given by the right multiplication by the matrix:

$$F_B := {}^t\begin{pmatrix} g_3 & 0 & 0 & 0 & 0 \\ 0 & g_3 & 0 & 0 & 0 \\ yB(I_d - g_1) & yB(I_d - g_2) & yB & I_d & 0 \\ 0 & 0 & 0 & g_3 & 0 \\ (g_3 - I_d)(g_1 - I_d) & (g_3 - I_d)(g_2 - I_d) & I_d - g_3 & 0 & I_d \end{pmatrix}.$$

6.2.3 A quotient vector space

The kernel vector bundle U has k linearly independent holomorphic sections given by the following formula, for $\underline{\breve{u}}_3 := (u_1, u_2, u_A, u_B) \in \mathrm{Im}(I_d - g_1) \times \mathrm{Im}(I_d - g_2) \times \mathbb{C}^{2d}$:

$$\sigma_{\underline{\breve{u}}_3} : (x, y) \mapsto \begin{pmatrix} u_1 \\ u_2 \\ -(g_1 g_2)^{-1}(u_1 + g_1 u_2 + ([B, A] - yB)u_A + (xBAB^{-1} - I_d)u_B) \\ u_A \\ u_B \end{pmatrix}.$$

Let us denote by Σ the $\mathbb{C}(x, y)$–vector space spanned by these sections ; right multiplication by the matrices F_1, F_2, F_A and F_B defined above gives us four linear automorphisms of this vector space through the following formulas, for $\sigma \in \Sigma$:

$$\begin{aligned}
F_i \cdot \sigma &: (x,y) \mapsto \sigma(x,y)F_i \in U_{(x,y)} \ (i \in [2]) \ ; \\
F_A \cdot \sigma &: (x,y) \mapsto \sigma\left(x, \frac{y}{c_3}\right) F_A \left(x, \frac{y}{c_3}\right) \in U_{(x, c_3^{-1} y)} \ ; \\
F_B \cdot \sigma &: (x,y) \mapsto \sigma\left(x, c_3 y\right) F_B \left(x, c_3 y\right) \in U_{(x, c_3 y)} \ .
\end{aligned}$$

Moreover, since $\gamma_1\gamma_2[\alpha,\beta] \in \Gamma_2$ acts as the conjugacy by $\alpha^{-1}\beta^{-1}\gamma_3\beta\alpha$ on Γ_3, $F_1F_2[F_A,F_B]$ acts on $GA_d(\mathbb{C})$ as the conjugacy by $((u_B, yB)\cdot(u_A, xA))^{-1}\cdot(u_3,g_3)\cdot(u_B,yB)\cdot(u_A,xA)$, which happens to have linear part equal to g_3 since the latter commutes to all of $GL_d(\mathbb{C})$. Therefore, again because g_3 is a scalar matrix, $F_1F_2[F_A,F_B]$ acts on $GA_d(\mathbb{C})$ as the conjugacy by a translation.

Therefore, a prerequisite to developing a working framework for our middle convolution will be to "kill" the diagonal action of $\mathbb{C}^d$ over U. More precisely, remark that for all $(x,y) \in \mathbb{C}^2$, $\underline{u} \in U_{(x,y)}$ and for all $v \in \mathbb{C}^d$, one gets:

$$(v,Id)\cdot((u_1,g_1),(u_2,g_2),(u_3,g_3),(u_A,A),(u_B,B))\cdot(-v,Id) = ((u_1+(1-g_1)v,g_1),\dots,(u_B+(1-B)v,B)) \ ,$$

therefore it makes sense to set

$$\psi(x,y) := \begin{pmatrix} I_d - g_1 \\ I_d - g_2 \\ I_d - g_3 \\ I_d - xA \\ I_d - yB \end{pmatrix}$$

and to consider the quotient $\mathbb{C}(x,y)$–vector space

$$E := \Sigma / V \ ,$$

where $V \subset \Sigma$ is the linear span $\langle \sigma_v \,|\, v \in \mathbb{C}^d \rangle$ with

$$1\sigma_v : (x,y) \mapsto \begin{pmatrix} (I_d - g_1)v \\ (I_d - g_2)v \\ (1 - c_3)v \\ (I_d - xA)v \\ (I_d - yB)v \end{pmatrix} .$$

The linear isomorphisms F_1 and F_2, act trivially on V, therefore they act on the quotient E but this not the case for F_A and F_B as they do not act trivially on the base. So we need to consider the subgroup

$$\langle F_1, F_2, F_A^r, F_B^r \rangle \leq \langle F_1, F_2, F_A, F_B \rangle$$

who does act trivially on V since $c_3^r = 1$. Therefore, one gets a group action of

$$\Gamma_2^r := \langle \gamma_1, \gamma_2, \alpha^r, \beta^r \rangle \leq \Gamma_2$$

on the quotient vector space E, whose dimension is:

$$\dim(E) = \dim(\Sigma) - d = \mathrm{rank}(I_d - g_1) + \mathrm{rank}(I_d - g_2) + d\,. \tag{E6.1}$$

The subgroup Γ_2^r has finite index equal to $2r$ in Γ_2; furthermore, it is isomorphic to the fundamental group of a complex torus with $4r$ punctures. Indeed, since it is an index $2r$ subgroup of Γ_2, there exists a $2r$–fold ramified cover X over the twice punctured torus such that $\Gamma_2^r \cong \pi_1(X)$ and the orbifold Riemann–Hurwitz formula ensures that X must be a $4r$ punctured complex torus.

6.2.4 Algorithm

Start with a group representation $\rho : \Gamma_2 \to SL_d(\mathbb{C})$ and let $h_i := \rho(\gamma_i)$ for $i \in [2]$, $A_0 := \rho(\alpha)$ and $B_0 := \rho(\beta)$; choose four non–zero complex numbers $\theta_1, \theta_2, \theta_\alpha$ and θ_β such that $c_3 := \theta_1\theta_2\theta_\alpha\theta_\beta \neq 1$ is a r–th root of unity. Now set $g_i := \theta_i h_i$ ($i \in [2]$), $g_3 := c_3 I_d$ and $A := \theta_\alpha A_0$, $B := \theta_\beta B_0$. We now have five matrices $g_1, g_2, g_3, A, B \in GL_d(\mathbb{C})$ satisfying the following conditions:

(CIE1) $g_1 g_2 g_3 [A, B] = I_d$, where $[A, B] = ABA^{-1}B^{-1}$;

(CIE2) $g_3 = c_3 I_d$ where $c_3 \in \mathbb{C}^* \setminus \{1\}$ is some r–th root of unity.

Considering the index $2r$ subgroup

$$\Gamma_2^r := \langle \gamma_1, \gamma_2, \alpha^r, \beta^r \rangle \leq \Gamma_2$$

isomorphic to the fundamental group of a torus with $4r$ punctures, we can define a (weaker) analog to Katz's middle convolution in this particular case.

Definition 6.2.1 (Virtual elliptic middle convolution)**.** *Suppose that we have ρ and $\underline{\theta}$ satisfying the above conditions. Let $\tilde{g}_1, \tilde{g}_2, \tilde{A}, \tilde{B}$ be the matrices of size*

$$d_\rho := \mathrm{rank}(I_d - g_1) + \mathrm{rank}(I_d - g_2) + d$$

corresponding to the action of F_1, F_2, F_A^r, F_B^r on the quotient vector space E defined in Section 6.2.3. Then we define the virtual elliptic middle convolution *of ρ with respect to $\underline{\theta}$ as the group representation*

$$\mathfrak{E}_{\underline{\theta}}(\rho) : \Gamma_2^r \to SL_{d_\rho}(\mathbb{C})$$

defined by

$$\gamma_1 \mapsto \frac{1}{\tau_1}\tilde{g}_1, \quad \gamma_2 \mapsto \frac{\tau_1\tau_A\tau_B}{c_3\tau_2}\tilde{g}_2, \quad \alpha \mapsto \frac{1}{\tau_A}\tilde{A}, \quad \beta \mapsto \frac{1}{\tau_B}\tilde{B}\,,$$

where $\tau_i^{d_\rho} = \det(\tilde{g}_i)$, $\tau_A^{d_\rho} = \det(\tilde{A})$, $\tau_B^{d_\rho} = \det(\tilde{B})$.

6.3 Effective computations

Now we describe an explicit `Maple` procedure to compute the virtual elliptic middle convolution of some quintuplet of matrices g_1, g_2, g_3, A, B satisfying conditions **(CIE1)** and **(CIE1)**. For the sake of simplicity, we will make the following extra assumptions:

- the entries are two by two matrices;
- $g_3 = -I_2$, *i.e* $r = 2$;
- 1 is an eigenvalue for both g_1 and g_2 , meaning that our output will be made up of four by four matrices.

This procedure requires the following set–up commands.

```
with(LinearAlgebra):
o:=ZeroMatrix(2):
Id := IdentityMatrix(2):
```

Start by setting up the fibre bundle U; to do so, we first compute the vector space $\mathrm{Im}(I_2 - g_1) \times \mathrm{Im}(I_2 - g_1)$:

```
ES:=op([ColumnSpace(Id-g1),  ColumnSpace(Id-g2)]);
vect :=simplify(Vector([op(convert(y1*ES[1][1],list)), op(
    convert(y2*ES[2][1],list)) ,x3,y3,xa,ya,xb,yb]));
```

and set up the matrix `M` corresponding to φ.

```
M := simplify(Matrix([Id ,g1, g1.g2,B.A.MatrixInverse(B).
    MatrixInverse(A)-y*B,x*B.A.MatrixInverse(B)-Id
          ]));
```

We know define the sections spanning Σ (here denoted by `se`i)

```
u1 := Vector[column]([ vect[1] ,
                   vect[2] ]):
u2 :=  Vector[column]([ vect[3] ,
                   vect[4] ]):  ua := Vector[column]([ xa,ya ]):
                      ub := Vector[column]([ xb,yb ]): u1,u2,ua,
                      ub;
u3 :=simplify( - MatrixInverse(g1.g2).( u1 + g1.u2 + (Com(B,A)
     - y*B).ua + (x*B.A.MatrixInverse(B)-Id).ub));
sigma := Vector[column]([ u1 ,
                   u2 ,
```

```
                  u3 ,
                  ua ,
                  ub ]);
se1 := subs(y1=1, y2=0, xa=0,ya=0,xb=0,yb=0,sigma):
se2 := subs(y1=0, y2=1, xa=0,ya=0,xb=0,yb=0,sigma):
se3 := subs(y1=0, y2=0, xa=1,ya=0,xb=0,yb=0,sigma):
se4:= subs(y1=0, y2=0, xa=0,ya=1,xb=0,yb=0,sigma):
se5 := subs(y1=0, y2=0, xa=0,ya=0,xb=1,yb=0,sigma):
se6 := subs(y1=0, y2=0, xa=0,ya=0,xb=0,yb=1,sigma):
```

and the matrices F_1, F_2, F_A^2 and F_B^2.

```
F1
:= Matrix([[ g1.(g3-Id)+Id , o , g1.(Id-g1), o , o ],
        [ (g2-Id).(g3-Id) , Id , (Id-g2).(g1-Id) , o , o ],
        [ Id-g3 , o , g1 , o , o ],
        [ o , o , o , Id , o ],
        [ o , o , o , o , Id ]]);
F2 := Matrix([[ Id ,o, o , o , o ],
        [ o , g2.(g3-Id)+Id , g2.(Id-g2) , o , o ],
        [ o , Id-g3 , g2 , o , o ],
        [ o , o , o , Id , o ],
        [ o , o , o , o , Id ]]);
Fa := Matrix([[ MatrixInverse(g3) , o , MatrixInverse(g3).(g1-
    Id) , o , o ],
        [ o , MatrixInverse(g3) , MatrixInverse(g3).(g2-Id) ,
            o , o ],
        [  o,  o, x*A , (Id-g3) , o ],
        [ o , o , o , Id , o ],
        [ o , o , -MatrixInverse(g3) , o , MatrixInverse(g3)
            ]]);
Fb:=Matrix([[ g3 , o ,y*(Id-g1).B , o , (g1-Id).(g3-Id) ],
        [ o , g3 , y*(Id-g2).B , o , (g2-Id).(g3-Id) ],
        [ o ,o  , y*B , o , Id-g3 ],
        [ o , o , Id , g3 , o ],
        [ o ,  o, o , o , Id ]]);
Fa2 := map(x->factor(x),Fa0.Fa0)
Fb2 := map(x->factor(x),Fb0.Fb0)
```

We then turn our attention to the subspace $V \subset \Sigma$; we represent the map ψ by the following matrix:

```
N:=simplify(Matrix([[ Id - g1], [Id - g2], [Id - g3], [Id -x*A], [Id - y*B]]));
```

In order to explicitly compute the action of the braids F_i on the quotient $E = \Sigma/V$ we need to find an adequate basis of Σ.

```
e1 := Vector[column]([ 1 ,
                 0 ]): e2 := Vector[column]([ 0 ,
                 1 ]):


sf1 := convert(simplify(Matrix([[ (Id - g1).e1],[(Id - g2).e1],[(Id - g3).e1],[(Id -x*A).e1],[(Id - y*B).e1]])),Vector): sf2 := convert(simplify(Matrix([[ (Id - g1).e2],[(Id - g2).e2],[(Id - g3).e2],[(Id -x*A).e2],[(Id - y*B).e2]])),Vector): sf1,sf2;
```

Assume for the sake of simplicity that the rank of the family (`se1`,`se2`,`se3`,`se4`,`sf1`,`sf1`) is six (up to changing one or more of the `se`i); now we can compute the virtual elliptic middle convolution of our initial quintuplet by applying the following procedure

```
CME := proc(F)
local sol1, sol2,sol3,sol4
simplify(F.se1-(x1*se1+x2*se2+x3*se3+x4*se4+x5*sf1+x6*sf2)): sol1:=solve(convert(%,set),{x1,x2,x3,x4,x5,x6}):
simplify(F.se2-(x1*se1+x2*se2+x3*se3+x4*se4+x5*sf1+x6*sf2)): sol2:=solve(convert(%,set),{x1,x2,x3,x4,x5,x6}):
simplify(F.se3-(x1*se1+x2*se2+x3*se3+x4*se4+x5*sf1+x6*sf2)): sol3:=solve(convert(%,set),{x1,x2,x3,x4,x5,x6}):
simplify(F.se4-(x1*se1+x2*se2+x3*se3+x4*se4+x5*sf1+x6*sf2)): sol4:=solve(convert(%,set),{x1,x2,x3,x4,x5,x6}):
RETURN(simplify(Transpose(Matrix([[ eval(x1,sol1) , eval(x1,sol2) , eval(x1,sol3),eval(x1,sol4) ],
        [ eval(x2,sol1) , eval(x2,sol2) , eval(x2,sol3),eval(x2,sol4)],
[ eval(x3,sol1) , eval(x3,sol2) , eval(x3,sol3),eval(x3,sol4)],
[ eval(x4,sol1) , eval(x4,sol2) , eval(x4,sol3),eval(x4,sol4)]])))):
end proc:
```

to the list below.

```
F := [F1,F2,Fa2,Fb2]:
```

Appendices

Appendix A

Computing mapping class group orbits

The purpose of this appendix is to describe the algorithm we used in various part of our work to compute the orbit of selected point of character varieties of punctured spheres. For the sake of simplicity we will give a complete overview of the five punctured case, as it is the most relevant to our study.

1.1 Pure mapping class group orbits

All of the following `Maple` procedures make use of the `LinearAlgebra` library. We start by defining a simple but useful auxiliary function giving the coordinates of the point in $\mathrm{Char}(0,5)$ associated with some list of five matrices `L`.

```
Ch:= proc(L)
RETURN(map(x->factor(x),[Trace(L[1]), Trace(L[2]), Trace(L[3]), Trace(L[4]),Trace(L[1].L[2]),Trace(L[1].L[3]),Trace(L[1].L[4]),Trace(L[2].L[3]),Trace(L[2].L[4]),Trace(L[3].L[4]),Trace(L[1].L[2].L[3]),Trace(L[1].L[2].L[4]),Trace(L[1].L[3].L[4]),Trace(L[2].L[3].L[4]),Trace(L[1].L[2].L[3].L[4])])): end proc:
```

Now recall that since we have the following isomorphism:

$$\mathrm{PMod}(0,5) \cong PB_4 \cong Z(PB_4)$$

we only need to compute the action of the pure braid group PB_4 on quintuplets of matrices, so we start by defining procedures realising the action of the elementary braids σ_i (and of their inverses) such that

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \,|\, [\sigma_1, \sigma_3],\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\, \sigma_3\sigma_2\sigma_3 = \sigma_2\sigma_3\sigma_2 \rangle \;.$$

```
s1 := proc(L)
RETURN([simplify(L[1].L[2].MatrixInverse(L[1])),L[1],L[3],L
    [4],L[5]]):
end proc:
s2 := proc(L)
RETURN([L[1],simplify(L[2].L[3].MatrixInverse(L[2])),L[2],L
    [4],L[5]]):
end proc:
s3 := proc(L)
RETURN([L[1],L[2],simplify(L[3].L[4].MatrixInverse(L[3])),L
    [3],L[5]]):
end proc:
s1i := proc(L)
RETURN([L[2],simplify(MatrixInverse(L[2]).L[1].L[2]),L[3],L
    [4],L[5]]):
end proc:
s2i := proc(L)
RETURN([L[1],L[3],simplify(MatrixInverse(L[3]).L[2].L[3]),L
    [4],L[5]]):
end proc:
s3i := proc(L)
RETURN([L[1],L[2],L[4],simplify(MatrixInverse(L[4]).L[3].L[4])
    ,L[5]]):
end proc:
```

This allows us to define procedures emulating the action of the generators of the pure braid group, namely, for $i \leq j < 3$:

$$\sigma_{i,j} := (\sigma_j \dots \sigma_{i+1})\sigma_i^2(\sigma_j \dots \sigma_{i+1})^{-1} .$$

```
F1 := proc(L)
RETURN(Ts3(Ts2(Ts1(Ts1(Ts2i(Ts3i(L))))))):
end proc:
F2 := proc(L)
RETURN(Ts3(Ts2(Ts2(Ts3i(L))))):
end proc:
F3 := proc(L)
RETURN(Ts3(Ts3(L))):
end proc:
F4 := proc(L)
```

```
RETURN(Ts2(Ts1(Ts1(Ts2i(L))))):
end proc:
F5 := proc(L)
RETURN(Ts2(Ts2(L))):
end proc:
F6 := proc(L)
RETURN(Ts1(Ts1(L))):
end proc:
F1i := proc(L)
RETURN(Ts3(Ts2(Ts1i(Ts1i(Ts2i(Ts3i(L))))))):
end proc:
F2i := proc(L)
RETURN(Ts3(Ts2i(Ts2i(Ts3i(L))))):
end proc:
F3i := proc(L)
RETURN(Ts3i(Ts3i(L))):
end proc:
F4i := proc(L)
RETURN(Ts2(Ts1i(Ts1i(Ts2i(L))))):
end proc:
F5i := proc(L)
RETURN(Ts2i(Ts2i(L))):
end proc:
F6i := proc(L)
RETURN(Ts1i(Ts1i(L))):
end proc:
```

Now, we set the following list to contain all generators of PB_4 and their inverses

```
F:=[F1,F2,F3,F4,F5,F6,F1i,F2i,F3i,F4i,F5i,F6i];
```

and use the following brute–force algorithm to compute the desired orbits.

```
Orb := proc(L)
local R,n,n0,k,i,M:
R := simplify([L]):
n0 := 0:
n := 1:
while n0<n do
for k from n0+1 to n do
for i from 1 to nops(F) do
M := map(x->factor(x),F[i](R[k])):
if not member(M,R) then
```

```
R:=[op(R),M]: fi:od:od:
n0 :=n :
n := nops(R):
od:
RETURN(n,R):
end proc:
```

Here, the entry `L` is assumed to be a quintuplet of matrices $M_1, \ldots, M_5$ such that the product $M_1 \cdots M_5$ is equal to the identity.

1.2 Mapping class group orbits

Now that we have an explicit way to compute orbits under the pure mapping class group $\mathrm{PMod}(0,5)$, it is quite straightforward to extend this procedure to the total mapping class group. Recall from [29] that the quotient $B_4/Z(b_4)$ is equal to the mapping class group of a four punctured disk, meaning it is isomorphic to the subgroup of $\mathrm{Mod}(0,5)$ stabilising a given puncture. As such, to compute the orbit of some point $[\underline{g}] \in \mathrm{Char}(0,5)$ (where $\underline{g}$ is a quintuplet of matrices such that $g_1 \cdots g_5 = I_2$) under the mapping class group of the five punctured sphere, we simply need to compute the orbits of every cyclic permutation of this quintuplet under the braid action, using the procedure below at each step.

```
OrbBraid := proc(L)
local R,n,n0,k,i,M:
R := simplify([L]):
n0 := 0:
n := 1:
while n0<n do
for k from n0+1 to n do
for i from 1 to nops(B4) do
M := map(x->factor(x),B4[i](R[k])):
if not member(M,R) then
R:=[op(R),M]: fi:od:od:
n0 :=n :
n := nops(R):
od:
RETURN(n,R):
end proc:
```

Here, `L` is assumed to be a list containing the quintuplet $\underline{g}$ and `B4` is the following list.

```
B4:=[Ts1,Ts2,Ts3,Ts1i,Ts2i,Ts3i];
```

Appendix B

Explicit computations relative to Chapters 3 and 4

This appendix is to be used as a companion to Chapters 3 and 4; we give here explicit `Maple` formulas for computing Garnier Hamiltonians, as defined in Section I1.2.

Suppose that we know all local exponents `K0`, `K1`, `Kt1`, `Kt2` and `Koo` of the system (these can be recovered from the monodromy data as explained in Section I1.2); then the following formulas give the Hamiltonians H_1, H_2 of the Garnier system $(\mathcal{G}_2)$

$$\begin{cases} \partial_{t_k}\mathbf{p}_i & = -\partial_{\mathbf{q}_i} H_k \quad i,k \in [2] \\ \partial_{t_k}\mathbf{q}_i & = \partial_{\mathbf{p}_i} H_k \quad\ \ i,k \in [2] \end{cases}.$$

First Hamiltonian.

```
H1:=-(1/4)*(4*t2*t1^2*q2^2*p2^2-4*p2^2*q2^4*q1+4*q1^2*k1*t1*q2
   *p1-4*t2*t1^2*q2*k0*p2
+4*q1^3*q2*p1+4*p2^2*q2^4*t1+4*p2^2*q2^3*q1-t1^2*q2+4*k0*t2*q1
   *q2*p2+4*k0*q1^3*t1*p1
-4*kt1*q1^3*q2*p1+4*kt2*q1^3*t1*p1-4*q1^2*t2*q2*p1-4*p2^2*q2
   ^3*t1^2-4*p2*q2^3*q1
-4*p2^2*q2^3*t1+q1*t1^2+4*kt1*t2*q1*q2*p2+4*q1^2*k1*t2*q2*p1
   -4*q1^2*k1*t1*t2*p1
+4*q1^2*t1*p1+4*p1^2*q1^2*t1*t2*q2-t1^2*q2*kt2^2+4*k0*t1*t2*q1
   *p1+4*k0*q1^2*t2*q2*p1
-4*q1^3*k1*q2*p1+4*q1^3*k1*t1*p1+4*p1^2*q1^3*t1*t2-4*p1^2*q1
   ^3*t2*q2+4*p1^2*q1^2*t2*q2
-4*p1^2*q1^3*t1*q2+4*p1^2*q1^2*t1*q2-4*k0*t1*t2*q2*p2+q1^2*q2
   +4*k0*t1*t2*q1*q2*p2
-4*k0*q1^3*q2*p1-4*kt2*q1^3*q2*p1+4*q1^2*t1*t2*p1-4*q1^2*kt2*
   t1*p1+4*q1^2*kt1*q2*p1
```

```
-4*q1^2*k0*t1*p1+4*k0*t1*q1*q2*p2+t1^2*q2*koo^2-t1^2*q2*k1^2-
    t1^2*q2*kt1^2-
4*p1^2*q1^3*q2-4*q1^2*q2*p1-4*p1^2*q1^4*t1+4*p1^2*q1^3*t1+4*
    kt2*t1*q1*q2*p2+4*q1*q2^2*p2
-4*k0*q1^2*t1*t2*p1+4*k0*q1^2*t1*q2*p1+4*kt1*q1^2*t2*q2*p1+4*
    kt2*q1^2*t1*q2*p1-4*t1^2*q2*k0*p2
+4*k0*t1*t2*q2*p1+4*t1^2*q2^2*p2^2-4*k1*t2*q1*q2^2*p2+4*k1*t1*
    t2*q2^2*p2-4*p2^2*q2^2*t1*t2*q1-
4*k1*t1*q1*q2^2*p2+4*kt2*q2^2*t1*p2-4*k0*q2^2*q1*p2+4*k1*q2^3*
    q1*p2-4*kt2*q2^2*q1*p2-2*q2^2*q1*kt2*k0
-2*q2^2*q1*kt2*kt1+4*q2^3*p2*t1-q2^2*q1-q2^2*q1*kt2^2-q2^2*q1*
    k0^2-q2^2*q1*kt1^2+4*k1*t1*t2*q1*q2*p2-4*k0*t1*t2*q1*p2+4*
    t1*q2^2*kt1*p2
+4*t1^2*q2^2*k1*p2+4*q2*p2*t1*t2-4*k0*t1*q1*q2*p1-4*k0*t2*q1*
    q2*p1+4*t2*t1*q2^2*kt1*p2+4*t2*t1^2*q1*k0*p1-4*t2*t1*q1^2*
    kt1*p1
-4*t2*t1^2*q1^2*p1^2+4*t1^2*q1*k0*p1-4*t1^2*q1^2*kt2*p1-4*t1
    ^2*q1^2*k1*p1-4*t1*q1^2*kt1*p1-4*t1^2*kt2*q2*p2-2*t1^2*q2*
    kt2*k0-2*t1^2*q2*kt2*kt1-2*t1^2*q2*kt2*k1+4*t1^2*q2^2*kt2*
    p2-4*kt2*t1*q1*q2*p1
-4*kt1*t2*q1*q2*p1+4*p1^2*q1^4*q2+4*q1*t1^2*kt2*p1+2*q1*t1^2*
    k0*k1+2*q1*t1^2*kt2*k0+2*q1*t1^2*kt2*kt1-4*p1^2*q1^2*t1*t2
    +2*t1^2*q2*kt2+2*t1^2*q2*k1+4*t2*q1*q2^2*p2-4*t2*q1*q2*p2-
    q1^2*t1*kt1^2+q1^2*t1*koo^2-q1^2*t1*k1^2+2*q1^2*t1*k0+2*q1
    ^2*t1*k1+2*q1^2*t1*kt1+2*q1^2*t1*kt2+q1^2*q2*kt2^2+q1^2*q2*
    k0^2+q1^2*q2*kt1^2-2*q1^2*t1*kt2*k0-
2*q1^2*t1*kt2*kt1-2*q1^2*t1*kt2*k1-2*q1^2*t1*k0*kt1-2*q1^2*t1*
    k0*k1-2*q1^2*t1*kt1*k1-q1^2*t1-q1^2*t1*kt2^2-q1^2*t1*k0
    ^2-4*k1*t1*t2*q1*q2*p1+q1*t1^2*kt2^2-2*q1*t1^2*k0-2*q1*t1
    ^2*k1-2*q1*t1^2*kt1+q1*t1^2*kt1^2+q1*t1^2*k1^2
-q1*t1^2*koo^2-2*t1^2*q2*k0*kt1-2*t1^2*q2*k0*k1-2*t1^2*q2*kt1*
    k1+2*t1^2*q2*kt1-q1^2*q2*koo^2+q1^2*q2*k1^2-2*q1^2*q2*k0-2*
    q1^2*q2*k1-2*q1^2*q2*kt1-2*q1^2*q2*kt2+2*q1^2*q2*kt2*k0+2*
    q1^2*q2*kt2*kt1+2*q1^2*q2*kt2*k1+2*q1^2*q2*k0*kt1+2*q1^2*q2
    *k0*k1+2*q1^2*q2*kt1*k1-4*t1^2*q1^2*k0*p1+2*t1^2*q2*k0-4*t1
    *t2*q1*p1
+4*k0*t1^2*q2^2*p2+q1*t1^2*k0^2-2*q1*kt2*t1^2+2*q1*t1^2*k0*kt1
    +2*q1*t1^2*kt2*k1+2*q1*t1^2*kt1*k1+4*t2*q1*q2*p1+4*q1^2*kt2
    *q2*p1+4*q1^2*k0*q2*p1-4*q2^2*p2*t1*t2-t1^2*q2*k0^2-4*q2^2*
    p2*t1
```

```
-4*p1^2*t1*t2*q1*q2-4*k0*t1*t2*q1*q2*p1+4*k0*q2^2*t1*t2*p2-4*
   k0*q2^2*t2*q1*p2-4*k0*q2^2*t1*q1*p2-4*kt1*q2^2*t2*q1*p2-4*
   kt2*q2^2*t1*q1*p2+4*p2^2*t1*t2*q1*q2
-4*kt1*q2^2*q1*p2+4*k0*q2^3*q1*p2+4*p2^2*q2^3*t1*q1+4*p2^2*q2
   ^3*t2*q1-4*k0*q2^3*t1*p2-4*kt2*q2^3*t1*p2+4*kt2*q2^3*q1*p2
   +4*kt1*q2^3*q1*p2+4*k0*q2^2*t1*p2-4*k1*t1*q2^3*p2+4*p2^2*q2
   ^2*t1*t2-4*p2^2*q2^2*t1*q1-4*p2^2*q2^2*t2*q1-4*p2^2*q2^3*t1
   *t2+t1*q2^2+4*t2*q1*t1^2*p1^2-2*q2^2*q1*kt2*k1-2*q2^2*q1*k0
   *kt1-2*q2^2*q1*k0*k1-2*q2^2*q1*kt1*k1+2*t1*q2^2*kt2*k0+2*t1
   *q2^2*kt2*kt1
+2*t1*q2^2*kt2*k1+2*t1*q2^2*k0*kt1+2*t1*q2^2*k0*k1+2*t1*q2^2*
   kt1*k1+q2^2*q1*koo^2-q2^2*q1*k1^2+2*q2^2*q1*k0+2*q2^2*q1*k1
   +2*q2^2*q1*kt1+2*q2^2*q1*kt2+t1*q2^2*kt2^2+t1*q2^2*k0^2+t1*
   q2^2*kt1^2-t1*q2^2*koo^2
+t1*q2^2*k1^2-2*t1*q2^2*k0-2*t1*q2^2*k1-2*t1*q2^2*kt1-2*t1*q2
   ^2*kt2+4*t2*t1^2*k0*p2
-4*t2*t1^2*k0*p1-4*q1^3*t1*p1+4*t2*q1*t1*kt1*p1+4*t2*q1*t1^2*
   k1*p1+4*t1*q1^3*kt1*p1-4*t2*t1^2*p2^2*q2-4*t1*q2^3*kt1*p2
   -4*t1^2*q1^2*p1^2+4*t1^2*q1^3*p1^2-4*t2*t1*kt1*q2*p2-4*t2*
   t1^2*k1*q2*p2)/((q1-q2)*(-t2+t1)*(-1+t1)*t1):
```

Second Hamiltonian.

```
H2:=(1/4)*(-4*p2^2*q2^4*q1+4*q1^2*k1*t1*q2*p1-2*t2*q2^2*kt1+4*
   p1^2*q1^3*t2-4*p2^2*q2^3*t2-2*t2*q2^2*k0+t2*q2^2*k0^2+4*q1
   ^3*q2*p1+4*p2^2*q2^3*q1-q1^2*t2*kt1^2+t2*q2^2*k1^2+4*k0*t2*
   q1*q2*p2-4*q1^2*t1*q2*p1-4*kt1*q1^3*q2*p1+4*kt1*q1^3*t2*p1
   +4*k0*q1^3*t2*p1-4*p2*q2^3*q1+q1*t2^2*k1^2+t2*q2^2*kt1^2-q1
   ^2*t2*kt2^2+t2*q2^2*kt2^2+4*kt1*t2*q1*q2*p2-
q1^2*t2+4*q1^2*k1*t2*q2*p1-4*q1^2*k1*t1*t2*p1+q1^2*t2*koo^2-q1
   ^2*t2*k1^2+2*q1^2*t2*k0+2*q1^2*t2*k1+2*q1^2*t2*kt1+2*q1^2*
   t2*kt2-t2*q2^2*koo^2+4*p1^2*q1^2*t1*t2*q2+4*k0*t1*t2*q1*p1
   +4*k0*q1^2*t2*q2*p1+4*q1^3*k1*t2*p1-4*q1^3*k1*q2*p1+4*p1^2*
   q1^3*t1*t2-4*p1^2*q1^3*t2*q2+4*p1^2*q1^2*t2*q2-4*p1^2*q1^3*
   t1*q2+4*p1^2*q1^2*t1*q2-
t2^2*q2*kt2^2-t2^2*q2*k0^2-t2^2*q2*kt1^2+t2^2*q2*koo^2-t2^2*q2
   *k1^2+2*t2^2*q2*k0+2*t2^2*q2*k1+2*t2^2*q2*kt2+4*q1^2*p1*t2
   -4*q1^2*p1^2*t2^2+4*k0*t1*t2^2*q1*p1-4*p1^2*q1^2*t1*t2^2-2*
   t2^2*q2*kt2*k1-2*t2^2*q2*k0*kt1-2*t2^2*q2*k0*k1-2*t2^2*q2*
   kt1*k1-4*k0*t1*t2^2*q2*p2-4*k0*t1*t2*q2*p2+q1^2*q2+4*t2*kt2
```

```
    *q2^2*t1*p2+4*k0*t1*t2*q1*q2*p2-4*k0*q1^3*q2*p1-4*kt2*q1^3*
    q2*p1+4*q1^2*t1*t2*p1+4*q1^2*kt1*q2*p1+4*k0*t1*q1*q2*p2-4*
    p1^2*q1^3*q2-4*p1^2*q1^4*t2-4*q1^2*q2*p1+4*kt2*t1*q1*q2*p2
    +4*q1*q2^2*p2-4*k0*q1^2*t1*t2*p1+4*k0*q1^2*t1*q2*p1
+4*kt1*q1^2*t2*q2*p1+4*kt2*q1^2*t1*q2*p1-2*t2*q2^2*k1+4*k0*t1*
    t2*q2*p1-q1^2*t2*k0^2-4*k1*t2*q1*q2^2*p2+4*k1*t1*t2*q2^2*p2
    -4*p2^2*q2^2*t1*t2*q1-4*k1*t1*q1*q2^2*p2+4*kt1*q2^2*t2*p2
    +4*k0*q2^2*t2*p2-4*k0*q2^2*q1*p2+4*k1*q2^3*q1*p2-4*k1*t2*q2
    ^3*p2-4*kt2*q2^2*q1*p2+2*t2*q2^2*kt2*k0+2*t2*q2^2*kt2*kt1
    +2*t2*q2^2*kt2*k1+2*t2*q2^2*k0*kt1+2*t2*q2^2*k0*k1+2*t2*q2
    ^2*kt1*k1-2*q2^2*q1*kt2*k0-2*q2^2*q1*kt2*kt1+4*q2^3*p2*t2
    -2*t2*q2^2*kt2-q2^2*q1-q2^2*q1*kt2^2-q2^2*q1*k0^2-q2^2*q1*
    kt1^2+4*k1*t1*t2*q1*q2*p2-4*k0*t1*t2*q1*p2+4*p2^2*q2^2*t1*
    t2^2+4*q2*p2*t1*t2-4*k0*t1*q1*q2*p1-4*k0*t2*q1*q2*p1
-t2^2*q2+q1*t2^2+4*p2^2*q2^4*t2-4*kt2*t1*q1*q2*p1-4*kt1*t2*q1*
    q2*p1+4*p1^2*q1^4*q2-4*p1^2*q1^2*t1*t2-2*q1^2*t2*kt2*k0-2*
    q1^2*t2*kt2*kt1-2*q1^2*t2*kt2*k1-2*q1^2*t2*k0*kt1-2*q1^2*t2
    *k0*k1-2*q1^2*t2*kt1*k1-4*t1*q1*q2*p2+4*t1*q1*q2^2*p2+q1^2*
    q2*kt2^2+q1^2*q2*k0^2+q1^2*q2*kt1^2-4*k1*t1*t2*q1*q2*p1-q1
    ^2*q2*koo^2+q1^2*q2*k1^2-2*q1^2*q2*k0-2*q1^2*q2*k1-2*q1^2*
    q2*kt1-2*q1^2*q2*kt2+q1*t2^2*k0^2+2*q1^2*q2*kt2*k0+2*q1^2*
    q2*kt2*kt1+2*q1^2*q2*kt2*k1+2*q1^2*q2*k0*kt1+2*q1^2*q2*k0*
    k1+2*q1^2*q2*kt1*k1-4*t1*t2*q1*p1+4*t1*q1*q2*p1+4*q1^2*kt2*
    q2*p1+4*q1^2*k0*q2*p1
-4*q1^2*k0*t2*p1-4*q1^2*kt1*t2*p1-4*q2^2*p2*t1*t2-4*t2*kt2*q2
    ^3*p2-4*q2^2*p2*t2-4*p1^2*t1*t2*q1*q2-4*k0*t1*t2*q1*q2*p1
    +4*k0*q2^2*t1*t2*p2-4*k0*q2^2*t2*q1*p2-4*k0*q2^2*t1*q1*p2
    -4*kt1*q2^2*t2*q1*p2-4*kt2*q2^2*t1*q1*p2+4*p2^2*t1*t2*q1*q2
    -4*p2^2*q2^3*t2^2-4*q1^3*p1*t2-4*q1^2*k0*t2^2*p1-4*q1^2*kt1
    *t2^2*p1+4*q1^3*kt2*p1*t2+4*p1^2*q1^3*t2^2-4*kt1*q2^2*q1*p2
    -4*k0*q2^3*t2*p2+4*k0*q2^3*q1*p2+4*p2^2*q2^3*t1*q1+4*p2^2*
    q2^3*t2*q1+4*kt2*q2^3*q1*p2+4*kt1*q2^3*q1*p2-4*kt1*q2^3*t2*
    p2+4*p2^2*q2^2*t1*t2-4*p2^2*q2^2*t1*q1-4*p2^2*q2^2*t2*q1-4*
    p2^2*q2^3*t1*t2+t2*q2^2-2*q1*t2^2*k1
-2*q2^2*q1*kt2*k1-2*q2^2*q1*k0*kt1-2*q2^2*q1*k0*k1-2*q2^2*q1*
    kt1*k1+q2^2*q1*koo^2-q2^2*q1*k1^2+2*q2^2*q1*k0+2*q2^2*q1*k1
    +2*q2^2*q1*kt1+2*q2^2*q1*kt2+4*k0*t1*t2^2*p2-2*t2^2*q2*kt2*
    k0-4*k0*t1*t2^2*p1-4*k0*t2^2*q2*p2-4*kt1*t2^2*q2*p2+4*k1*t2
    ^2*q2^2*p2+4*t2*kt2*q2^2*p2-4*p2^2*t1*t2^2*q2-4*q1^2*kt2*t1
    *p1*t2+4*p2^2*q2^2*t2^2+4*q1*kt2*t1*p1*t2+4*q1*k1*t1*t2^2*
```

```
   p1
-2*q1*t2^2*k0-2*q1*t2^2*kt2-q1*t2^2*koo^2-2*q1*t2^2*kt1+q1*t2
   ^2*kt2^2+q1*t2^2*kt1^2+2*q1*t2^2*kt1*k1+4*q1*p1^2*t1*t2
   ^2+2*q1*t2^2*kt2*k0+2*q1*t2^2*kt2*kt1+2*q1*t2^2*kt2*k1+2*q1
   *t2^2*k0*kt1+2*q1*t2^2*k0*k1+4*q1*k0*t2^2*p1+4*q1*kt1*t2^2*
   p1-4*q1^2*kt2*p1*t2-4*q1^2*k1*t2^2*p1-4*k1*t1*t2^2*q2*p2+4*
   kt1*q2^2*t2^2*p2+4*k0*q2^2*t2^2*p2-4*t2*kt2*t1*q2*p2+2*kt1*
   t2^2*q2-2*t2^2*q2*kt2*kt1)/(t2*(q1-q2)*(-1+t2)*(-t2+t1)):
```

Appendix

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Résumé

Une déformation isomonodromique d'une sphère épointée est une famille de connexions logarithmiques plates sur cette dernière ayant toutes, à conjugaison globale près, la même représentation de monodromie. Ces objets sont paramétrés par les solutions d'une certaine famille d'équations aux dérivées partielles, les systèmes de Garnier, qui sont équivalents dans le cas de la sphère à quatre trous aux équations de Painlevé VI. L'objet des travaux présentés ici est de construire de nouvelles solutions algébriques des ces systèmes dans le cas de la sphère à cinq trous. Dans une première partie, nous classifions les déformations isomonodromiques algébriques obtenues par restriction aux droites d'une connexion logarithmique plate sur $\mathbb{P}^2(\mathbb{C})$ *dont le lieu polaire est une courbe quintique. On obtient ainsi deux nouvelles familles de solutions algébriques du système de Garnier associé. Dans une deuxième partie, nous exploitons le fait qu'une déformation isomonodromique algébrique correspond à une orbite finie sous l'action du groupe modulaire sur la variété des caractères de la sphère à cinq trous pour obtenir de nouveaux exemples de telles orbites. Nous employons pour ce faire la convolution intermédiaire sur les représentations de groupes libres développée par Katz. Enfin, nous décrivons une généralisation partielle de ce procédé au cas d'un tore complexe à deux trous.*

Abstract

We call isomonodromic deformation any family of logarithmic flat connections over a punctured sphere having the same monodromy representation up to global conjugacy. These objects are parametrised by the solutions of a particular family of partial differential equations called Garnier systems, which are equivalent to the Painlevé VI equations in the four punctured case. The purpose of this thesis is to construct new algebraic solutions of these systems in the five punctured case. First, we give a classification of algebraic isomonodromic deformations obtained by restricting to lines some logarithmic flat connection over $\mathbb{P}^2(\mathbb{C})$ *whose singular locus is a quintic curve. We obtain two new families of algebraic solutions of the associated Garnier system. In a second part, we use the fact that any algebraic isomonodromic deformation corresponds to a finite orbit under the mapping class group action on the character variety of the five punctured sphere to obtain new examples of such orbits. We do this by using Katz's middle convolution on representations of free groups. Finally, we give a partial generalisation of this procedure in the case of a twice punctured complex torus.*