Limits theorems for some functionals of Markov chains with heavy tails: convergence to stable process with index $\alpha \in (0, 2)$

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Contents

Stable distributions/Processes with index α, Definitions

Stable distributions

A random variable (r.v.) X has a stable law if there exists constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that:
$X_1 + \ldots + X_n \overset{d}{=} c_n X + d_n \quad \forall n \geq 1$, where X_i, $i \geq 1$ are i.i.d copy of X.

- If $d_n = 0$ one says that X is strictly stable.
- For such a distribution, it is known that $c_n = cst\, n^{1/\alpha}$ with $\alpha \in]0, 2]$.
- α is called the index of stability.
- if $\alpha = 2$, X is the gaussian r.v.
- We'll say that X is an α-stable r.v or simply α-stable.

The Lévy Khintchine representation

- In the case $\alpha \in (0,2)$ one specify the characteristic function of an α-stable random variable X as follows $\mathbb{E}e^{i\theta X} = e^{\psi(\theta)}$ where

$$\psi(\theta) = ib\theta + \int (e^{i\theta x} - 1 - i\theta x 1_{|x|\leq 1}) \frac{1}{|x|^{1+\alpha}} (c_1 1_{x<0} + c_2 1_{x>0}) dx,$$

  with $c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. We'll say that X is α-stable with characteristics (b, c_1, c_2).
- Other representations exists,[Samorodnisky and Taqqu], [Zoloarev] but their are the same.
- In the sequel, an α-stable process $(X_t, t \geq 0)$ is a Lévy process such that the r.v. X_1 is α-stable.

Contents

The generalized Central Limit Theorem (GCLT)

GCLT, [Gnedenko-Kolmogorov]

Let $(X_i)_{i\geq 1}$ be iid random variables with common cumulative distribution function F. Denotes by Y_α an α-stable with local caracteristics (c_1, c_2). There exists suitable constants $a_n > 0$, $b_n \in \mathbb{R}$ such that

$$a_n^{-1} \sum_{i=1}^{n} X_i - b_n \xrightarrow{d} Y_\alpha$$

if and only if $G(x) = 1 - F(x) - F(-x)$ is regularly varying of order $-\alpha$ and
$\lim_{x\to\infty} \frac{1-F(x)}{1-F(x)+F(-x)} = \frac{c_1}{c_1+c_2}$, $\lim_{x\to\infty} \frac{F(-x)}{1-F(x)+F(-x)} = \frac{c_2}{c_1+c_2}$

Example

- The Pareto distribution X of type I, with tail distribution $\mathbb{P}[X > x] = c\, x^{-\alpha} \mathbb{1}_{x>0}$ with $c > 0$ and $\alpha \in (0, 2)$, belongs
- Let $Z \sim \Gamma(a, \lambda)$, where $\Gamma(a, \lambda)$ is the Gamma distribution with parameters a and $0 < \lambda < 2$. Define $X = e^Z$. We have $\mathbb{P}[X > x] \sim y^{-\lambda} L(y)$ for some slowly varying function L so that $X \in \mathcal{D}(\alpha, Y_\alpha)$.
- (a_n) can be taken to be $a_n = n^{1/\alpha}$
- and if $\alpha \in (1, 2)$ we can take $b_n = n\mathbb{E}(X)$ and $b_n = 0$ for $\alpha \in (0, 1)$.
- If $\alpha = 2$, this result improves in particular the CLT.

Contents

Mixing sequences

- Let $\mathcal{F}_j$ (resp. $\mathcal{G}_j$)) be respectively the past and the future σ-fields generated by a sequence (X_n) for $0 \leq n \leq j$ (resp.$j \leq n$).

Strongly Mixing [Bradley, Rio]

The strong mixing coefficient $\alpha(n)$ is defined as :

$$\alpha(n) = \sup_j \{\sup_{A,B} (\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)) \; A \in \mathcal{F}_j, \; B \in \mathcal{G}_{j+n}\}.$$

- If $\lim_{n\to\infty} \alpha(n) = 0$, the sequence is strongly mixing.

ρ-Mixing and φ-Mixing [Peligrad, Doukhan]

The ρ-mixing coefficient $\rho(n)$ is defined as the maximal correlation coefficient, i.e.

$$\rho(n) = \sup_j \{ \sup_{F,G} \; Corr(F, G) \, , \; F \in L^2(\mathcal{F}_j) \, , \; G \in L^2(\mathcal{G}_{n+j}) \} \, .$$

The φ-mixing coefficient $\varphi(n)$ is defined as:

$$\varphi(n) = \sup_j \{ \sup_{A,B} \left(\mathbb{P}(B|A) - \mathbb{P}(B) \right) \, , \; A \in \mathcal{F}_j \, , \; B \in \mathcal{G}_{j+n} \} .$$

- If $\lim_{n\to\infty} \rho(n) = 0$ the sequence is ρ-mixing.
- If $\lim_{n\to\infty} \varphi(n) = 0$, the sequence is φ-mixing.

Quantitative ergodicity for Markov chains

- Let (X_n) be an irreducible, aperiodic and positive recurrent Markov chain with invariant probability measure μ. We denote by P the transition kernel of the chain.

Rates of ergodicity

For $p, r \geq 1$ we define

$$\alpha_{r,p}(n) = Sup \left\{ \| P^n g \|_{L^p(\mu)} , \forall g : \| g \|_{L^r(\mu)} = 1 \text{ and } \int g \, d\mu = 0 \right\}$$

We define similarly $\alpha^*_{p,r}(n)$ for the adjoint operator P^*.

- Interestings rates in the sequel are the following: $\alpha_{2,2}$ or $\alpha_{p,p}$, $\alpha_{\infty,2}$ and $\alpha_{1,\infty}$

- $\alpha_{2,2}$ or $\alpha^*_{2,2}$ is given by the following well known spectral gap estimate:

Proposition

Under the previous assumptions for the chain, the following are equivalent

(1) $\forall f \in \mathbb{L}^2(\mu)$ and all n, $Var_\mu(P^n f) \le e^{-2\lambda\, n}\, Var_\mu(f)$,

(2) there exists $C \ge 1$, such that $\forall f \in \mathbb{L}^2(\mu)$,

$$Var_\mu(f) \le C \, \langle (I - P^* P) f, f \rangle := \int (f - P^* P f)\, f \, d\mu ,$$

where C is given by the relation $e^{-\lambda} = \frac{C-1}{C}$.

- This shows that $\alpha_{2,2}(n) \le e^{-\lambda n}$ for f such that $\mu(f) = 0$ and $||f||_2 = 1$. It follows also that $\alpha_{p,p}(n) \le 2 e^{-\lambda n 2 \frac{(p-1)}{p}}$ for all $p > 1$.

Example

- Consider the Birth and death chain (BDC) in the space $\mathbb{N}$

$$P(x, x+1) = p_x > 0,\ P(x, x-1) = q_x > 0,$$

$P(x,x) = 1 - p_x - q_x,\ q_0 = 0$
The chain is positively recurrent if and only if

$$\sum_{x>0} \frac{p_0\, p_1 \ldots p_{x-1}}{q_1 \ldots q_x} := \sum_{x>0} \lambda_x < +\infty\,,$$

in which case, the unique invariant (reversible) distribution is given by $\mu(x) = \mu(0)\, \lambda_x$ for all $x > 0$.

- It is well known that there exist a spectral gap provided

$$q_x \geq \frac{(d+1)b}{b-1} + bp_x \quad \text{for some } b > 1 \ \text{ and } \ d > 0$$

Example

- The reflected random walk on $\mathbb{N}$ is given from (BDC) when p_x and q_x are constant:

  $$p_x = p, \quad q_x = q = 1 - q, \quad \text{and} \quad P(x,x) = 0 \text{ if } \quad x \neq 0$$

  The chain is positive recurrent if $q > p$ and the invariant distribution (reversible) is the geometric distribution on $\mathbb{N}$

  $$\mu(dx) = (1 - \frac{p}{q})(\frac{p}{q})^x$$

- The above spectral gap condition for (BDC) enforces the parameters to be $q > p$ for the reflected random walk.

About $\alpha_{\infty,2}$.

- It can be compute in terms of weak Poincaré inequalities. We call weak Poincaré inequality (WPI), an inequality of the form:

$$Var_{\mu}(f) \leq \beta(s)\langle (I - P^*P)f, f\rangle + s||f - \mu(f)||_{\infty}^{2} \quad s \in (0,1] \quad (\spadesuit),$$

where β is a non increasing function.

- In the continuous time case, but using derivatives, this was introduced by **M. Röckner and F.Y. Wang**.

Proposition

- If the (WPI) in (♠) holds for some non increasing function β, then $\alpha_{\infty,2}(n)$ and $\alpha^*_{\infty,2}(n)$, $\to 0$ as $n \to \infty$.
- Conversely, if P is symetric, any decay to 0 of $\alpha_{\infty,2}(n)$ will imply some (WPI) in (♠).

In general however, we do not know wether $\alpha_{\infty,2}$, $\alpha^*_{\infty,2}$ are equal or have the same behaviour.

$$\text{About} \quad \alpha_{1,\infty} \quad \text{or} \quad \alpha^*_{1,\infty}.$$

- The rate $\alpha_{1,\infty}(n)$ specifies that the semi groupe $(P_n, n \geq 0)$ is hyperbounded (hypercontractivity or ultracontractivity) and can be described in terms of log-Sobolev inequalities.

Link between Mixing property and ergodicity property.

Proposition

For all n, let $[n]$ denotes the integer part of n.

(1) $\alpha^2_{\infty,2}(n) \vee (\alpha^*)^2_{\infty,2}(n) \leq 4\alpha(n) \leq \alpha_{\infty,2}([n/2])\, \alpha^*_{\infty,2}([n/2])$.

(2)
$$\alpha^2_{2,2}(n) = (\alpha^*)^2_{2,2}(n) \leq \rho(n) \leq \; c\, \alpha_{2,2}(n)\,.$$

(3) $\varphi(n) \leq \alpha^2_{1,\infty}([n/2])\,.$

Idea of the proof

- Use equivalents definitions of Mixing coeficients in terms of covariance representations, and the Markov property.

Contents

Definition

- We shall say that a random variable Z_1 is regularly varying of index $\alpha > 0$ if there exist some $c \in [0,1]$ such that for all $x > 0$,

$$\lim_{u \to +\infty} \frac{\mathbb{P}(Z_1 > ux)}{\mathbb{P}(||Z_1| > u)} = cx^{-\alpha}, \quad \lim_{u \to +\infty} \frac{\mathbb{P}(Z_1 < -ux)}{\mathbb{P}(|Z_1| > u)} = (1-c)x^{-\alpha}$$

- If Z_1 is regularly varying, there exists a slowly varying function L such that for all $x > 0$,

$$\mathbb{P}(|Z_1| > x) = x^{-\alpha} L(x)\,.$$

- Note that for $\alpha \in (0,2)$, this allows to deal with non square integrable random variables.

Limits theorems for stationary regularly and mixing sequences

Let $(Z_i)_{i\geq 1}$ be a stationary and regularly varying sequences and denotes by S_n, the partial sum sequences

- When dealing with convergence to stable distributions/processes, we need some kind of centering , namely we shall consider:
$$T_n = \frac{S_n - nc_n}{a_n}, \; T_n(t) = \frac{S_{[nt]} - ntc_n}{a_n}$$
where $c_n = \mathbb{E}[Z_1 1_{|X_1| \leq a_n}]$ and a_n is defined by $\lim_{n\to\infty} n\mathbb{P}[|Z|_1 > a_n] = 1$.
- It is interesting to look at the asymptotic behavior of
$$\frac{nc_n}{a_n} = \frac{n}{a_n}\mathbb{E}\left[Z_1 \, \mathbb{I}_{|Z_1|\leq a_n}\right]$$

Proposition

For $\alpha \in (1,2)$ and $\mathbb{E}_\mu(Z_1) = 0$ we have

$$\lim_{n \to +\infty} \frac{n\, c_n}{a_n} = \frac{\alpha}{1-\alpha}\,(2c-1)\,.$$

- The same relation holds in the case $\alpha \in (0,1)$ (thanks to Karamata Theorem).
- This proposition allows to deal only with $S_{[nt]}$ up to some slight change in the characteristics of the limit process.

About the litterature in the context of Mixing sequences

- (1)[Davis] **R.Davis** Stable limits for partial sums of dependent random variables, Ann.Probab. n11, 262-269, (1983).
- (2) [TK] **M.Tyran-Kaminska**. Convergence to Lévy stable processes under some weak dependence conditions. Stochastic Process. Appl. 120, n9, 1629-1650, (2010)
- (3) [BKS] **B.Basrak, D.Krizmanic, and J.Seger**. A functional limit theorem for dependent sequences with infinite variance stable limits. Ann.Probab. 40, N5 2008-2033 (2012).
- (4) [BJMW] **K.Bartkiewicz and Al**. Stable limits for sums of dependent infinite variance random variables. Probab.Theory Relat.Fields, 150: 337-372, (2011).

- The following condition called the **Anti-clustering conditions** (AC) which is familar from extreme value theory, is relevant when dealing with convergence to α-stable process.

Sufficient Anti Clustering Condition (AC)

$$\lim_{k\to+\infty}\ \limsup_{n\to+\infty} \sum_{j=2}^{[n/k]} n\,\mathbb{P}_{\mu}\left(|Z_1| > \varepsilon\, a_n\,,\, |Z_j| > \varepsilon\, a_n\right) = 0, \quad \forall\, \varepsilon > 0.$$

- Such conditions are assumed in the above papers, namely known as : condition $\mathcal{D}'$ of **R.Davis**, condition (x) of **Denker and Jakubowski**.

Necessary Anti Clustering Condition (NAC)

A neccessary Anti clustering condition is described by the following condition: (Condition (3.19) in [**M.Tyran-Kaminska**])

$$\lim_{n\to+\infty} \mathbb{P}_{\mu}\left(\max_{2\leq j\leq r(n)} |Z_j| > \varepsilon\, a_n \,\big|\, |Z_1| > \varepsilon\, a_n\right) = 0\,. \quad r(n) = o(n)$$

- In the previous we discussed spectral gap properties and compared them to mixing properties.
- Now, once this correspondance is understood, the only thing to do, is to check the so called "Anti clustring" conditions.

Contents

Theorem

Let $(X_n)_{n\geq 0}$ be an irreducible, aperiodic and positive recurrent Markov chain with unique invariant probability measure μ. Let f be such that $Z_1 = f(X_1)$ is regularly varying of index $\alpha \in (1,2)$ and $\int f d\mu = 0$.

1. *Assume that the chain has a spectral gap.*
2. *If $P|f|$ or $P^*|f|$ belongs to $\mathbb{L}^{\alpha+\beta}(\mu)$ for some $\beta > 0$ then*

$\frac{1}{a_n} S_{[nt]}(f) = \frac{1}{a_n} \sum_{j=1}^{[nt]} f(X_j)$ converges (under $\mathbb{P}_\mu$), in the Skorohod topology to an α-stable process.

- A CLT of this kind was obtained in 2009 by **[M.Jara, T.Komorowski and S.Olla]** "Limits Theorems for additive functionals of a Markov chain". Ann. of Applied Probab., 19(6): 2270-2300, (2009) using martingale approximation.

The next result contains the case $\alpha \in (0,1]$.

Theorem

Let $(X_n)_{n\geq 0}$ be an irreducible, aperiodic and positive recurrent Markov chain with unique invariant probability measure μ. Let f be such that $Z_1 = f(X_1)$ is regularly varying of index $\alpha \in (0,2)$. Assume that f is symetric under $\mathbb{P}_\mu$ in the case $\alpha = 1$.

1. *Assume also that the chain has a spectral gap.*
2. *If for some $\eta \leq \alpha$, $f \in \mathbb{L}^\eta(\mu)$ and $P|f|^\eta$ or $P^*|f|^\eta$ belongs to $\mathbb{L}^{1+\beta}(\mu)$ for some $\beta > 0$ such that $\eta > \frac{\alpha}{(1+\beta)}$, then*

$\frac{1}{a_n} S_{[nt]}(f) = \frac{1}{a_n} \sum_{j=1}^{[nt]} f(X_j)$ converges (under $\mathbb{P}_\mu$), in the Skorohod topology to an α-stable process.

Contents

Example

- For the reflected random walk, the Necessary Anti clustering Condition (NAC) fail to holds.
- A nice example of Markov chain in superdiffusion of energy in a lattice dynamic was studied by **[Jara,Olla,Komorowski]**.

Example

One can also consider the following situation:
$P = P_T$ for some $T > 0$, where $(P_t)_{t\geq 0}$ is continuous time hyperbounded Markov process with invariant measure π.

- In this case π will satisfy a log-Sobolev inequality with $I - \frac{1}{2}(P + P^*)$ as infinitesimal generator.
- and we can look at numerical schemes (Euler scheme) for approximating hypercontractive diffusion processes
- Note that in many situations the hypercontractivity property is preserved.