Meteotsunamis and Proudman resonance

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Outline

Tsunami

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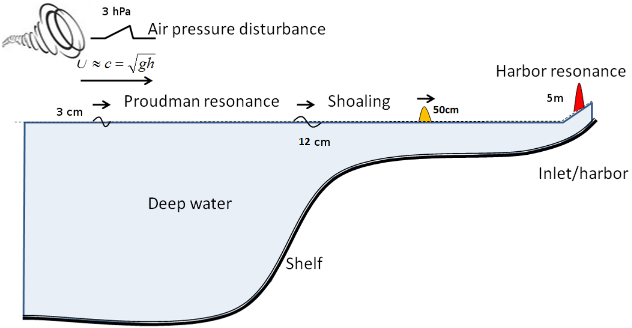
- Amplitude offshore : 50 centimeters.

Meteotsunami



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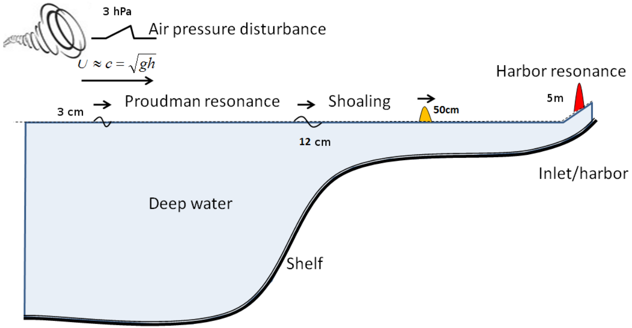

One example : Nagasaki(1979), "Abiki", pressure disturbance traveled at 22-39 m/s. Amplitude on the coast : 5m.

Mechanisms

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- We consider a disturbance which traveling at the speed U.
- The typical speed of long ocean waves $c = \sqrt{gH}$ where H is the typical depth.
- There is a resonance if U is close to c.

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- The water depth is bounded from below by $h_{\min} > 0$.

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- We consider the following standard problem :

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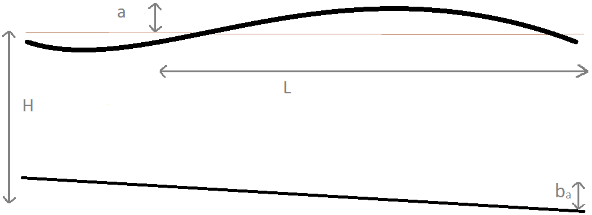
- If $\zeta = 0$ and $b=0$, $\widehat{G[\zeta,b](\psi)}(\xi) = |\xi|\tanh(H|\xi|)\widehat{\psi}(\xi)$.

We can reformulate the whole problem with the two unknown variables (ζ, ψ) :

$$\begin{cases} \partial_t \zeta - G[\zeta, b](\psi) = 0 \\ \partial_t \psi + \zeta + \frac{1}{2}|\nabla_x \psi|^2 - \frac{(G[\zeta,b](\psi)+\nabla_x \zeta . \nabla_x \psi)^2}{2(1+|\nabla_x \zeta|^2)} = -\frac{P_0}{\rho}. \end{cases}$$

Relevant parameters



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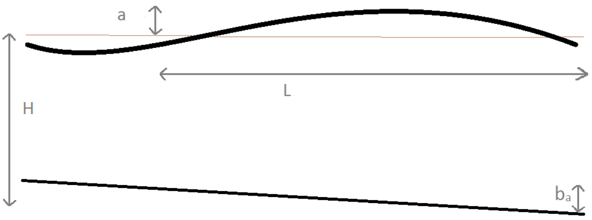

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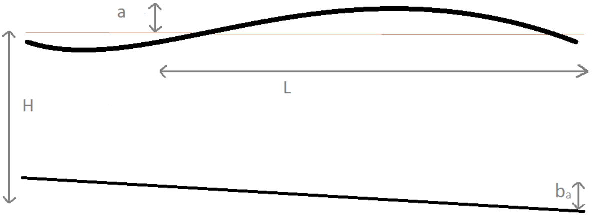

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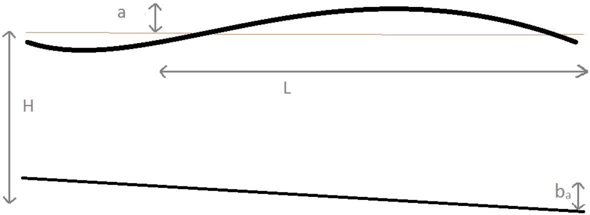

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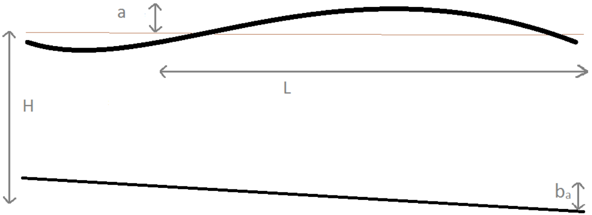

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- The typical speed : $c = \sqrt{gH}$ (in the shallow-water case).

We can renormalize the variables :

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- $L \sim 30\text{-}100km$, $H \sim 50\text{-}500m$ and $a \sim 3\text{-}20cm$.
- $\varepsilon \sim 10^{-3}$, $\mu \sim 10^{-5}\text{-}10^{-6}$, $c \sim 22\text{-}70m/s$.

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and the nondimensionalized Zakharov formulation :

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G[\varepsilon\zeta, \beta b](\psi) = 0 \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla_x \psi|^2 - \varepsilon(\cdots) = -P_0. \end{cases}$$

Rayleigh-taylor criterion

We can compute the pressure P in Ω_t. We denote by $\mathfrak{a}(\zeta,\psi) := -(\partial_z P)_{|z=\varepsilon\zeta}$, called the Rayleigh-Taylor coefficient.

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$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} + \begin{pmatrix} \varepsilon \underline{V}_0 \cdot \nabla & \frac{1}{\mu} G_\mu[\varepsilon\zeta_0, \beta b](\cdot) \\ \mathfrak{a}(\zeta_0,\psi_0) & \varepsilon \underline{V}_0 \cdot \nabla \end{pmatrix} \begin{pmatrix} \zeta \\ \psi \end{pmatrix} + \text{lower term} = \begin{pmatrix} 0 \\ -P \end{pmatrix}$$

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If we denote by $A := \begin{pmatrix} i\varepsilon \underline{V}_0 \cdot \xi & \frac{1}{\mu} G_\mu \widehat{[\varepsilon\zeta_0, \beta b]} \\ \mathfrak{a}(\zeta_0, \psi_0) & i\varepsilon \underline{V}_0 \cdot \xi \end{pmatrix}$, we need that $Sp(A) \subset i\mathbb{R}$ (hyperbolicity).

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Therefore, we need that $\mathfrak{a}(\zeta_0,\psi_0) > 0$.

Existence and uniqueness of the Water waves equation

For $s \geq 0$, we define $\dot{H}^s(\mathbb{R}^d) := \{u \in L^2_{\text{loc}}(\mathbb{R}^d), \nabla u \in H^{s-1}(\mathbb{R}^d)\}$.

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Theorem

Let N large enough.
Let $P \in \mathcal{C}^0(\mathbb{R}^+, H^N(\mathbb{R}^d))$, $(\psi_0, \zeta_0) \in \dot{H}^N(\mathbb{R}^d) \times H^N(\mathbb{R}^d)$, such that

$$1 + \varepsilon\zeta_0 + \beta b_0 \geq h_{\min} \text{ and } \underline{\mathfrak{a}}(\psi_0, \zeta_0) \geq \mathfrak{a}_0.$$

There exists $T > 0$, $(\psi, \zeta) \in \mathcal{C}^0\left(\left[0, \frac{T}{\max(\varepsilon,\beta)}\right], \dot{H}^N(\mathbb{R}^d) \times H^N(\mathbb{R}^d)\right)$ unique solution of the water waves equation with the initial data (ψ_0, ζ_0).

An asymptotic model

We suppose that ε, β, μ are small. We have

$$\begin{cases} \partial_t \zeta - \overbrace{\frac{1}{\mu} G[\varepsilon\zeta, \beta b](\psi)}^{\sim -\partial_x^2 \psi} = 0 \\ \partial_t \psi + \zeta + \underbrace{\frac{\varepsilon}{2} |\nabla_x \psi|^2 - \varepsilon\,(\cdots)}_{\sim \varepsilon C(|\zeta|_{H^N}, |b|_{H^N}) \|\psi\|_{\dot{H}^N}} = -P_0. \end{cases}$$

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Proposition

For N large enough, there exists $T > 0$ and a solution of the water waves equation $(\psi, \zeta) \in \mathcal{C}^0\left(\left[0, \frac{T}{\max(\varepsilon,\beta)}\right], \dot{H}^{N+1}(\mathbb{R}^d) \times H^N(\mathbb{R}^d)\right)$,

$$\begin{cases} \partial_t \zeta + \partial_x^2 \psi = \max(\varepsilon, \beta, \mu) R, \\ \partial_t \psi + \zeta = -P_0 + \max(\varepsilon, \beta, \mu) S. \end{cases}$$

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with $P_0 = P(x - Ut)$ and $U > 0$.

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- And the landslide-generated tsunamis ?