There are no fractional Brownian fields indexed by the cylinder

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Plan of the talk

Gaussian random field

Definition

Given any set T, we call a *Gaussian random field* indexed by T any collection of Gaussian random variables $(X_t)_{t \in T}$ such that any linear combination of these is also a Gaussian random variable.

Fractional Brownian fields

Definition

Given a metric space (E, d) and $H > 0$, we call *an H-fractional Brownian field indexed by E* any Gaussian field $(X_t)_{t \in E}$ such that

- $\forall t \in E$, $\mathbb{E}\, X_t = 0$
- $\forall s, t \in E$, $\mathbb{E}(X_s - X_t)^2 = [d(s,t)]^{2H}$

- If we consider $E = \mathbb{R}$ the classical fractional Brownian motion meet the definition.
- Such fields enjoy nice properties like stationary increments regarding the isometry group of E and *often* local $H-$self similarity.

Examples of index sets (E,d)

- $\mathbb{R}^d$, or any Hilbert space,
- $\mathbb{S}^d$,
- a discrete metric space (with a graph structure for example),
- a Riemannian manifold.

Examples of index sets (E,d)

- $\mathbb{R}^d$, or any Hilbert space, for $0 < H \leq 1$,
- $\mathbb{S}^d$, for $0 < H \leq 1/2$,
- a discrete metric space (with a graph structure for example),
- a Riemannian manifold.

Alas the existence of fractional Brownian fields depends on the space (E, d) and is in general not easy to check.

A precision

- The above definition is not enough to guaranty unicity (in law) of the field. Indeed for any centred Gaussian variable N, if (X_t) is an H-fractional Brownian field, then so is $(X_t + N)$.

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- We can choose an arbitrary origin point $O \in E$ and ask that $X_O = 0$ a.s. . The covariance is then

$$\mathbb{E}\, X_s X_t = 1/2 \left(d^{2H}(O, s) + d^{2H}(O, s) - d^{2H}(s, t) \right)$$

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  and since the field is Gaussian it is unique in law.
- We will work with one of those fields from now on.

The existence problem

- Because an H-fractional Brownian field is a Gaussian field, it exists if and only if its covariance is a positive definite kernel. Schoenberg's theorem implies that it is the case if and only if d^{2H} is a kernel of negative type, that is to say

Definition

for all $n \in \mathbb{N}^*$, $P_1 \cdots P_n \in E$ and $c_1, \cdots, c_n$ such that $\sum_{i=1}^n c_i = 0$,

$$\sum_{i,j=1}^{n} c_i c_j d^{2H}(P_i, P_j) \leq 0.$$

- Observe that the origin O have disappeared. Furthermore one can check that all the fractional Brownian fields exists at once.

Fractional exponent of a metric space

Theorem (Gangolli 67)

Given a metric space (E, d), there exists $0 \leq H_E \leq \infty$ such that

$$d^{2H} \text{ is of negative type } \Leftrightarrow H \leq H_E.$$

Idea of proof : x^α is a Bernstein function for $0 < \alpha \leq 1$.

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- $H_{\mathbb{H}^d} = 1/2$ (existence via direct Lévy-Chentsov geometrical construction) ,
- $H_{ellipsoid} < H_{sphere} = 1/2$,
- For any Riemannian manifold M with at least one point of strictly positive curvature $H_M < 1$.

Lévy Khinchin formula

- On symmetric spaces G/K one can use harmonic analysis to characterise continuous kernels of negative type.
- It gives a characterisation of the continuous K-invariants kernels ψ of negative type on G

$$\psi(x) = Q(x) + \int [1 - \mathrm{Re}\omega(x)] \, \mathrm{d}\mu(\omega),$$

which is not very practical in general.
- If G is compact Gangolli gave a criterion in term of sign of L^2 scalar product with elementary spherical functions, and derived $H_{\mathbb{S}^d} = 1/2$.

The result

Theorem

- *For all* $H > 0$*, there exists no fractional Brownian motion indexed by* $\mathbb{S}^1 \times]0, \varepsilon[$ *endowed with its geodesic distance. In other terms,* $H_{\mathbb{S}^1 \times]0,\varepsilon[} = 0$.

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- *The results holds for any metric space with a subset isometric to $\mathbb{S}^1 \times]0, \varepsilon[$, such as $\mathbb{S}^1 \times N$ with N any Riemannian manifold.*

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- *The results holds for any metric space with a subset isometric to* $\mathbb{S}^1\times]0,\varepsilon[$*, such as* $\mathbb{S}^1 \times N$ *with* N *any Riemannian manifold.*

- Other examples of metric space with fractional exponent equal to 0 are quaternionic hyperbolic spaces and $\mathbb{R}^n$ endowed with the norms $\left(\sum_{i=1}^n (x_i)^q\right)^{1/q}$ for $n \geq 3, q > 2$.

Sketch of the proof

- Let us call any tuple $(P_1, \cdots, P_N)$ in a metric space together with coefficients $(c_1, \cdots, c_N)$ such that $\sum_{i=1}^{N} c_i = 0$ *a configuration of points*.
- We will exhibit for all $H > 0$ a sequence $(P_1^N, \cdots, P_N^N), (c_1^N, \cdots, c_N^N)$ of configurations of points in the cylinder such that

$$\lim_{N\to\infty} \sum_{i,j=1}^{N} c_i^N c_j^N d^{2H}(P_i^N, P_j^N) = +\infty,$$

which will prove that d^{2H} is not a kernel of negative type.

A configuration on the circle

Let us consider a circle of perimeter 1, parametrized by arc length, and take the points $(P_i)_{i=1}^{4N}$ of coordinates $(i/4N)_{i=1}^{4N}$, together with the coefficients $c_i = (-1)^i$. We are interested in

$$A_N := \sum_{i,j=1}^{4N} c_i c_j \left[d_{\mathbb{S}^1}(P_i, P_j)\right]^{2H}.$$

Lemma

$$A_N \underset{N\to\infty}{\sim} C(H) N^{1-2H}$$

Duplicating the circle configuration, 1/2

- Consider the cylinder $\mathbb{S}^1 \times \mathbb{R}$ with d its classical geodesic distance.
- Take $8N$ points $(P_i)_{i=1}^{8N}$ such that $(P_i)_{i=1}^{4N} = (i/4N, 0)_{i=1}^{4N}$ and $(P_{i+4N})_{i=1}^{4N} = (i/4N, z_N)_{i=1}^{4N}$, with $z_N > 0$, and set again $c_i = (-1)^i$.
- We call

$$C_N = \sum_{i,j=1}^{8N} c_i c_j \left[d(P_i, P_j)\right]^{2H}$$

$$\begin{aligned}
&= \sum_{i,j=1}^{4N} c_i c_j [d(P_i, P_j)]^{2H} + \sum_{i,j=4N+1}^{8N} c_i c_j [d(P_i, P_j)]^{2H} \\
&+ \sum_{i=1}^{4N} \sum_{j=4N+1}^{8N} c_i c_j [d(P_i, P_j)]^{2H} + \sum_{i=4N+1}^{8N} \sum_{j=1}^{4N} c_i c_j [d(P_i, P_j)]^{2H} \\
&= 2A_N + 2B_N(z_N)
\end{aligned}$$

Duplicating the circle configuration, 2/2

Lemma

$\forall\ 0 < \alpha_1 < \alpha_2,\ \exists\ u_N \to 0,\ \forall\ z_N > 0$ *such that*

- $z_N = o\left(N^{-\alpha_1}\right)$
- $N^{-\alpha_2} = o\left(z_N\right)$

then

$$B_N(z_N) = \frac{H}{4^{H-1}} + u_N.$$

Mulitiplying the circle configuration, 1/2

- We choose $0 < \beta < \gamma < 1$ and consider the $\lfloor N^\beta \rfloor + 1$ circles at heights $\frac{k}{N^\gamma}$ for $k \in \{0, \cdots, \lfloor N^\beta \rfloor\}$.
- We put on the k-th of these circles $4N$ points $(P_i^k)_{i=1}^{4N}$ of coordinates $\left(i/4N, \frac{k}{N^\gamma}\right)_{i=1}^{4N}$, associated to our usual coefficients $c_i^k = (-1)^i$.
- We now focus on

$$Q_N = \sum_{k,l=0}^{\lfloor N^\beta \rfloor} \sum_{i,j=1}^{4N} c_i c_j \; d^{2H}(P_i^k, P_j^l)$$

$$= \left(\lfloor N^\beta \rfloor + 1\right) A_N + \sum_{k,l=0, k \neq l}^{\lfloor N^\beta \rfloor} B_N \left(z_N^{k,l}\right),$$

with $z_N^{k,l} = \frac{|k-l|}{N^\gamma}$.

Mulitiplying the circle configuration, 2/2

Applying our Lemma we get

$$
\begin{aligned}
Q_N &= \left(\lfloor N^\beta \rfloor + 1\right) A_N + \sum_{k,l=0,k\neq l}^{\lfloor N^\beta \rfloor} \frac{H}{4^{H-1}} + o(1) \\
&= \left(\lfloor N^\beta \rfloor + 1\right) A_N + \frac{\lfloor N^\beta \rfloor \left(\lfloor N^\beta \rfloor - 1\right)}{2} \left(\frac{H}{4^{H-1}} + o(1)\right)
\end{aligned}
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$$= \left(\lfloor N^\beta \rfloor + 1\right) A_N + \frac{\lfloor N^\beta \rfloor \left(\lfloor N^\beta \rfloor - 1\right)}{2} \left(\frac{H}{4^{H-1}} + o(1)\right)$$

Recall that $A_N \underset{N\to\infty}{\sim} C(H)N^{1-2H}$, therefore if we choose $\beta > 1 - 2H$ we obtain

$$Q_N \underset{N\to\infty}{\sim} \frac{\lfloor N^\beta \rfloor \left(\lfloor N^\beta \rfloor - 1\right)}{2} \left(\frac{H}{4^{H-1}}\right) \underset{N\to\infty}{\longrightarrow} +\infty \text{ as we wanted.}$$

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