Characters of the dihedral group

Let $n \geq 3$. Our aim is to determine the characters of the dihedral group $D_n := \langle r, s \mid s^2 = r^n = \mathrm{id}, srs = r^{-1} \rangle$. We will at first assume n to be even.

Unidimensional representations. Let ψ be a one–dimensional representation of D_n. Then $\psi(r)^n = \psi(r^n) = 1$, thus $\psi(r) \in \mu_n(\mathbb{C})$. We also have $\psi(s) \in \{-1, 1\}$ and $\psi(srs) = \psi(s)^2\psi(r) = \psi(r)$ therefore $\psi(r) = \psi(r)^{-1}$ ergo $\psi(r)^2 = 1$. In conclusion $\psi(r) \in \{-1, 1\}$. So there are 4 one–dimensional characters :

	s	r
χ_1	1	1
χ_2	1	-1
χ_3	-1	1
χ_4	-1	-1

Or :

	r^k	sr^k
χ_1	1	1
χ_2	1	-1
χ_3	$(-1)^k$	$(-1)^k$
χ_4	$(-1)^k$	$(-1)^{k+1}$

Two–dimensional representations. Let $\omega := e^{2i\pi/n}$ and let $h \in \mathbb{Z}$. We can define a two–dimensional representation ρ^h (whose character we denote by χ^h) of D_n as follows :

$$\rho^h(r^k) := \begin{pmatrix} \omega^{hk} & 0 \\ 0 & \omega^{-hk} \end{pmatrix} \text{ and } \rho^h(sr^k) := \begin{pmatrix} 0 & \omega^{-hk} \\ \omega^{hk} & 0 \end{pmatrix}$$

I.e ρ^h is defined as the only homomorphism $D_n \to GL_2(\mathbb{C})$ such that :

$$\rho^h(r) := \begin{pmatrix} \omega^{h} & 0 \\ 0 & \omega^{-h} \end{pmatrix} \text{ and } \rho^h(s) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

One can easily verify that ρ^h only depends on $h \mod n$; moreover, the representations ρ^h and ρ^{n-h} are isomorphic. Indeed, if one sets $T := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ one has $T\rho^h(r) = \rho^{-h}(r)T = \rho^{n-h}(r)T$ and $T\rho^h(s) = \rho^{-h}(s)T = \rho^{n-h}(s)T$. Therefore, one can assume $0 \leq h \leq \dfrac{n}{2}$ when studying any such representation.

- $h = 0$. One easily checks that $\chi^0 = \chi_1 + \chi_2$. Therefore, ρ^0 is isomorphic to the direct sums of the representations associated with χ_1 and χ_2 and is thus reducible.
- $h = \dfrac{n}{2}$. Since $\omega^{n/2} = e^{i\pi} = -1$ we have $\chi^{n/2} = \chi_3 + \chi_4$ and so $\rho^{n/2}$ is reducible.
- $0 < h < \dfrac{n}{2}$. Remark that is this case, $\omega^h \neq \omega^{-h}$, therefore the only lines of $\mathbb{C}^2$ which are stable under $\rho^h(r)$ are the canonical axes $\langle(1, 0)\rangle$ and $\langle(0, 1)\rangle$. But these subspaces are not stable under $\rho^h(s)$ hence ρ^h doesn't have any nontrivial subrepresentation and so is irreducible. Its character is then given by :

	s	r
χ^h	0	$2\cos\left(\dfrac{2h\pi}{n}\right)$

Or :

	r^k	sr^k
χ^h	$2\cos\left(\frac{2hk\pi}{n}\right)$	0

Now let us suppose that there exists $T \in GL_2(\mathbb{C})$ and $h \neq h'$ ($0 < h, h' < \frac{n}{2}$) such that $T\rho^h = \rho^{h'}T$. Then $T\rho^h(r)T^{-1} = \rho^{h'}(r)$ so $\rho^h(r)$ and $\rho^{h'}(r)$ have the same eigenvalues, i.e $\omega^h = \omega^{h'}$ which is absurd. Therefore, the representations ρ^k for $0 < k < \frac{n}{2}$ are not isomorphic.
Since $|D_n| = 2n = 4 \times 1 + \left(\frac{n}{2} - 1\right) \times 2^2$, we have successfully classified the isomorphism classes of D_n's characters.

If n is odd, we only have two unidimensional representations :

	s	r
χ_1	1	1
χ_2	1	-1

One then concludes as previously that all the ρ^h for $0 < h \leq \frac{n-1}{2}$ are irreducible and not isomorphic. Since $2n = 2 \times 1 + \frac{n-1}{2} \times 2^2$ one has again successfully accounted for all isomorphism classes.

References

[Ser98] Jean-Pierre Serre. *Représentations linéaires des groupes finis.* Hermann, 1998.